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# Dispersion Relations for $\pi \rightarrow \pi \pi$ 1

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In the following, we want to confirm some of the calculations from <1210.0793> (Hoferichter et al.), in particular regarding the dispersion relations therein.

We start with Eq.(5), which can be derived by transforming into the CMS (See Mathematica),

$$q = \begin{pmatrix} E_q \\ \vec{q} \end{pmatrix}, \quad p_1 = \begin{pmatrix} E_{p_1} \\ -\vec{q} \end{pmatrix}, \quad p_2 = \begin{pmatrix} E_{p_2} \\ \vec{p} \end{pmatrix}, \quad p_0 = \begin{pmatrix} E_{p_0} \\ -\vec{p} \end{pmatrix},$$

$$E_q = \frac{s - M_\pi^2}{2\sqrt{s}}, \quad E_{p_1} = \frac{s + M_\pi^2}{2\sqrt{s}}, \quad |\vec{q}| = \frac{s - M_\pi^2}{2\sqrt{s}},$$

$$E_{p_2} = \frac{\sqrt{s}}{2} = E_{p_0}, \quad |\vec{p}| = \frac{\sqrt{s}}{2} \sigma_\pi(s), \quad \sigma_\pi(s) = \sqrt{1 - \frac{4M_\pi^2}{s}},$$

where we define  $(-\vec{q}) \cdot \vec{p} = |\vec{q}| \cdot |\vec{p}| \cos \theta$ , i.e.  $\theta$  is the scattering angle between  $\vec{p}$  and  $(-\vec{q})$ . Then  $(s = (q + p_1)^2 = (p_2 + p_0)^2$

$$t = \frac{3M_\pi^2 - s}{2} + \frac{s - M_\pi^2}{2} \sigma_\pi(s) \cos \theta,$$

$$t = (p_1 - p_2)^2 = (p_0 - q)^2$$

$$u = (p_1 - p_0)^2 = (p_2 - q)^2$$

$$u = \underbrace{\frac{3M_\pi^2 - s}{2}}_{= a_s} - \underbrace{\frac{s - M_\pi^2}{2} \sigma_\pi(s) \cos \theta}_{= b_s}.$$

Due to isospin symmetry, the scalar function  $F(s, t, u)$  of

$$\mathcal{M}(s, t, u) = i E_{\text{prop}} E^T p_1^\nu p_2^\alpha p_0^\beta F(s, t, u) (\delta(q) \pi^-(p_1) \rightarrow \pi^-(p_2) \pi^0(p_0))$$

is fully symmetric in its arguments. Hence, the partial-wave decomposition reads

$$z = \cos \theta$$

$$F(s, t, u) = \sum_{\text{odd } \ell} f_\ell(s) P_\ell^1(z),$$

$$G_{\ell+} \cdot G_{\ell-} = G_{\ell+} \cdot G_{\ell-} \Rightarrow G_\ell = -1 \Rightarrow I_2 = 0 \Rightarrow I_{\text{initial}} = 1$$

$\Rightarrow$   $\ell$  odd in final state due to Bose symmetry

When the P-wave  $f_1(s)$  can be projected out using

$$f_1(s) = \frac{3}{4} \int_{-1}^1 dz (1-z^2) F(s, t, u).$$

In the absence of inelastic contributions, its imaginary part reads

$$\text{Im} f_1(s) = \sigma_s^\pi (t_1'(s))^* f_1(s) \theta(s - 4M_\pi^2),$$

with the  $\pi\pi$  P-wave amplitude  $t_1'(s) = \frac{e^{i\delta(s)} - 1}{2i\sigma_\pi(s)}$  ;

This unitarity relation looks similar to the ones we derived in the other notes and we save for ourselves a proof here.

(and in particular the  $\pi\pi$  VFF one)

Note that the unitarity relation implies Watson's final-state theorem, namely that the phase of  $f_1(s)$  coincides with  $\delta_1^+(s)$ .

Neglecting the imaginary parts of partial waves with  $l \geq 3$ , the amplitude may be decomposed as

$$F(s, t, u) = F(s) + F(t) + F(u).$$

Is this splitting only possible because  $F(s, t, u)$  is symmetric in  $s, t, u$ ?

Following the referenced paper <0702213>, we will now obtain once- and twice-subtracted dispersion relations for  $F(s, t, u)$  and consequently  $F(s)$  (see also Melnik's Master thesis for more on fixed- $t$  dispersion relations, which we will make use of here). To this end, we let  $t$  be fixed ( $u = 3M_\pi^2 - s - t$ ) and consider the fixed- $t$  dispersion relations

$$F(s, t) = C_1(t) + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im} F(s', t)}{s'(s'-s)} + \frac{u}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im} F(s', t)}{s'(s'-u)},$$

$$F(s, t) = C_2(t) + \frac{s^2}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im} F(s', t)}{s'^2(s'-s)} + \frac{u^2}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im} F(s', t)}{s'^2(s'-u)},$$

where the respectively second integral covers the left-hand cut in the variable  $s$  (i.e. right-hand cut in  $u$ ).

# Dispersion Relations for $\mathcal{F} \rightarrow \text{part 2}$

or another subtraction constant  $C_2(s)$

Here, there cannot be a term linear in  $s$  in the subtraction constant  $C_2(t)$ , since  $\mathcal{F}(s,t)$  is symmetric under  $s \leftrightarrow u$ . The  $t$ -dependence can be obtained from the symmetry under  $s \leftrightarrow t$ ,  $\mathcal{F}(0,t) = \mathcal{F}(t,0)$ , i.e.

$$C_1(t) + \frac{3M^2 - t}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',t)}{s'(s' - 3M^2 + t)}$$
$$= C_1(0) + \frac{t}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',0)}{s'(s' - t)} + \frac{3M^2 - t}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',0)}{s'(s' - 3M^2 + t)}$$

$$C_2(t) + \frac{(3M^2 - t)^2}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',t)}{s'^2(s' - 3M^2 + t)}$$
$$= C_2(0) + \frac{t^2}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',0)}{s'^2(s' - t)} + \frac{(3M^2 - t)^2}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',0)}{s'^2(s' - 3M^2 + t)}$$

$$\Rightarrow C_1(t) = C_1(0) + \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',0)}{s'} \left( \frac{t}{s' - t} + \frac{3M^2 - t}{s' - 3M^2 + t} \right)$$
$$- \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',t)}{s'} \frac{3M^2 - t}{s' - 3M^2 + t}$$

$$C_2(t) = C_2(0) + \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',0)}{s'^2} \left( \frac{t^2}{s' - t} + \frac{(3M^2 - t)^2}{s' - 3M^2 + t} \right)$$
$$- \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',t)}{s'^2} \frac{(3M^2 - t)^2}{s' - 3M^2 + t}$$

Inserting this into the initial expressions for  $\mathcal{F}(s,t)$  yields

$$\mathcal{F}(s,t) = C_1(t) + \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',t)}{s'} \left( \frac{s}{s' - s} + \frac{u}{s' - u} \right)$$

$$= C_1(0) + \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',0)}{s'} \left( \frac{t}{s' - t} + \frac{3M^2 - t}{s' - 3M^2 + t} \right)$$

$$- \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',t)}{s'} \frac{3M^2 - t}{s' - 3M^2 + t}$$

$$+ \frac{1}{\pi} \int_{4M^2}^{\infty} ds' \frac{\text{Im} \mathcal{F}(s',t)}{s'} \left( \frac{s}{s' - s} + \frac{u}{s' - u} \right)$$

$$F(s, t) = G_2(t) + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} F(s', t)}{s'^2} \left( \frac{s^2}{s' - s} + \frac{u^2}{s' - u} \right)$$

$$= G_2(t) + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} F(s', 0)}{s'^2} \left( \frac{t^2}{s' - t} + \frac{(3M^2 - t)^2}{s' - 3M^2 + t} \right) - \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} F(s', t)}{s'^2} \frac{(3M^2 - t)^2}{s' - 3M^2 + t} + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} F(s', t)}{s'^2} \left( \frac{s^2}{s' - s} + \frac{u^2}{s' - u} \right)$$

Note that these dispersion relations already exhibit symmetry under  $s \leftrightarrow u$  for fixed  $t$ , whereas symmetry under  $s \leftrightarrow t$  for fixed  $u$  is not manifest (yet). In order to impose this symmetry, we expand the absorptive part of  $F(s', t)$  in partial waves and write this according to

$$\text{Im} F(s', t) = \text{Im} f_l(s') + \text{Im} \Phi(s', t),$$

where  $\text{Im} \Phi(s', t)$  contains the higher partial waves with  $l \geq 3$ , i.e.

$$\text{Im} \Phi(s', t) = \sum_{\text{odd } l \geq 3} \text{Im} f_l(s') P_l'(\cos \theta).$$

With this decomposition, we can write the dispersion relations as

$$F(s, t) = G_1(t) + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} f_l(s') + \text{Im} \Phi(s', 0)}{s'} \left( \frac{t}{s' - t} + \frac{3M^2 - t}{s' - 3M^2 + t} \right) - \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} f_l(s') + \text{Im} \Phi(s', t)}{s'} \frac{3M^2 - t}{s' - 3M^2 + t} + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} f_l(s') + \text{Im} \Phi(s', t)}{s'} \left( \frac{s}{s' - s} + \frac{u}{s' - u} \right) = G_1(t) + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} f_l(s')}{s'} \left( \frac{s}{s' - s} + \frac{t}{s' - t} + \frac{u}{s' - u} \right) + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} \Phi(s', t)}{s'} \left( \frac{s}{s' - s} + \frac{u}{s' - u} - \frac{3M^2 - t}{s' - 3M^2 + t} \right) + \frac{1}{\pi} \int_{\frac{4M^2}{4k^2}}^{\infty} ds' \frac{\text{Im} \Phi(s', 0)}{s'} \left( \frac{t}{s' - t} + \frac{3M^2 - t}{s' - 3M^2 + t} \right), \quad \left. \begin{array}{l} (1) \\ (2) \end{array} \right\}$$

## Dispersion relations for $l \rightarrow \infty$ 3

$$\begin{aligned}
 F(s,t) &= C_2(t) + \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{\text{Im} f_1(s') + \text{Im} \Phi(s',0)}{s'^2} \left( \frac{t^2}{s'-t} + \frac{(3m^2-t)^2}{s'-3m^2+t} \right) \\
 &\quad - \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{\text{Im} f_1(s') + \text{Im} \Phi(s',t)}{s'^2} \frac{(3m^2-t)^2}{s'-3m^2+t} \\
 &\quad + \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{\text{Im} f_1(s') + \text{Im} \Phi(s',t)}{s'^2} \left( \frac{s^2}{s'-s} + \frac{u^2}{s'-u} \right) \\
 &= C_2(t) + \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{\text{Im} f_1(s')}{s'^2} \left( \frac{s^2}{s'-s} + \frac{t^2}{s'-t} + \frac{u^2}{s'-u} \right) \\
 &\quad + \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{\text{Im} \Phi(s',t)}{s'^2} \left( \frac{s^2}{s'-s} + \frac{u^2}{s'-u} - \frac{(3m^2-t)^2}{s'-3m^2+t} \right) \\
 &\quad + \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{\text{Im} \Phi(s',0)}{s'^2} \left( \frac{t^2}{s'-t} + \frac{(3m^2-t)^2}{s'-3m^2+t} \right).
 \end{aligned}$$

The part (1) - and similar for the twice-subtracted version - is now fully symmetric in  $s \leftrightarrow t \leftrightarrow u$ , whereas this symmetry is not manifest in (2). (Note that the above equations are called Roy equations.)

In the approximation of neglecting the imaginary parts of partial waves with  $l \geq 3$ , we thus have the dispersion relations

$$\begin{aligned}
 F(s,t,u) &= C_1 + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'} \left\{ \frac{s}{s'-s} + \frac{t}{s'-t} + \frac{u}{s'-u} \right\} \text{Im} f_1(s') \\
 &= C_2 + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'^2} \left\{ \frac{s^2}{s'-s} + \frac{t^2}{s'-t} + \frac{u^2}{s'-u} \right\} \text{Im} f_1(s'). \quad (*)
 \end{aligned}$$

By virtue of  $F(s,t,u) = F(s) + F(t) + F(u)$ , these dispersion relations correspond to

$$F(s) = \frac{C_1}{3} + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'} \frac{s}{s'-s} \text{Im} F(s')$$

replace  $\text{Im} f_1(s)$  by  $F(s)$  because it's a dispersion relation or  $F(s)$  or  $\text{Im} f_1(s) = \text{disc} F(s) = \frac{1}{3} (C_2^{(1)} + C_2^{(2)} s) + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'^2} \frac{s^2}{s'-s} \text{Im} F(s')$ ,  
 we already here need  $f_1(s) = F(s) + \Phi(s)$ !?

since  $F(s)$  has no symmetry that would exclude a term

linear in  $s$ , as it was the case for  $F(s, t)$ , so that a dispersive representation should contain such a term; here,

$$C_2^{(1)} + C_2^{(2)} 4\mu^2 = C_2, \quad (*)$$

given that  $s+t+u = 34\mu^2$  and  $\mathcal{F}(s, t, u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$ .

30.04.2021

Defining  $f_1(s) = \mathcal{F}(s) + \hat{\mathcal{F}}(s)$ , where  $\hat{\mathcal{F}}(s)$  is real on the RHC and

$$\hat{\mathcal{F}}(s) = \mathcal{B} \langle (1-z^4)\mathcal{F} \rangle, \quad \langle z^4\mathcal{F} \rangle = \frac{1}{2} \int_{-1}^1 dz z^n \mathcal{F}(t),$$

(see the other notes for a derivation), elastic unitarity yields

$$\text{Im} f_1(s) = \text{Im} \mathcal{F}(s) = (\mathcal{F}(s) + \hat{\mathcal{F}}(s)) \rho(s - 4\mu^2) \text{si} \hat{\sigma}_1(s) e^{-i\hat{\sigma}_1(s)}$$

The solution of this equation is given by (see the other notes)

$$\begin{aligned} \mathcal{F}(s) &= \Omega(s) \left\{ C_0 + \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s') \text{si} \hat{\sigma}_1(s')}{s'(s'-s) |\Omega(s')|} \right\} \\ &= \Omega(s) \left\{ \bar{C}_0^{(1)} + \bar{C}_0^{(2)} s + \frac{s^2}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s') \text{si} \hat{\sigma}_1(s')}{s'^2 (s'-s) |\Omega(s')|} \right\}. \end{aligned}$$

We match this to the representation from before by requiring

$$\mathcal{F}(0) \stackrel{!}{=} \frac{C_1}{3} \Rightarrow C_0 = \frac{C_1}{3}$$

$$\mathcal{F}(0) \stackrel{!}{=} \frac{C_2^{(1)}}{3}, \quad \hat{\mathcal{F}}(0) \stackrel{!}{=} \frac{C_2^{(2)}}{3} \Rightarrow \bar{C}_0^{(1)} = \frac{C_2^{(1)}}{3}$$

$$\begin{aligned} \hat{\mathcal{F}}(s) &= \Omega(s) \left\{ \bar{C}_0^{(1)} + \bar{C}_0^{(2)} s + \frac{s^2}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s') \text{si} \hat{\sigma}_1(s')}{s'^2 (s'-s) |\Omega(s')|} \right\} \\ &+ \Omega(s) \left\{ \bar{C}_0^{(2)} + \frac{s^2}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s') \text{si} \hat{\sigma}_1(s')}{s'^2 (s'-s)^2 |\Omega(s')|} + \frac{2s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s') \text{si} \hat{\sigma}_1(s')}{s'^2 (s'-s) |\Omega(s')|} \right\} \end{aligned}$$

$$\Rightarrow \hat{\mathcal{F}}(0) = \Omega(0) \bar{C}_0^{(1)} + \Omega(0) \bar{C}_0^{(2)} \stackrel{!}{=} \frac{C_2^{(2)}}{3}$$

$$\Rightarrow \bar{C}_0^{(2)} = \frac{C_2^{(2)}}{3} - \Omega(0) \bar{C}_0^{(1)} = \frac{C_2^{(2)}}{3} - \frac{C_2^{(1)}}{3} \Omega(0).$$

We thus have

$$\begin{aligned} \mathcal{F}(s) &= \Omega(s) \left\{ \frac{C_1}{3} + \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s') \text{si} \hat{\sigma}_1(s')}{s'(s'-s) |\Omega(s')|} \right\} \\ &= \Omega(s) \left\{ \frac{C_2^{(1)}}{3} (1 - \Omega(0)s) + \frac{C_2^{(2)}}{3} s + \frac{s^2}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s') \text{si} \hat{\sigma}_1(s')}{s'^2 (s'-s) |\Omega(s')|} \right\} \end{aligned}$$

# Dispersion Relations for $\pi \rightarrow \pi \pi$ 4

We now note that

$$\begin{aligned}\Omega(s) &= \exp \left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1'(s')}{s'(s'-s)} \right\} \\ \text{evaluated above} \\ \text{the cut} &\rightarrow \exp \left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1'(s')}{s'(s'-s)} + \frac{s}{\pi} i\pi \frac{\delta_1'(s)}{s} \right\} \\ &= \exp \left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1'(s')}{s'(s'-s)} \right\} e^{i\delta_1(s)} \\ &= |\Omega(s)| e^{i\delta_1(s)},\end{aligned}$$

so that

$$f_1(s) = \mathcal{F}(s) + \hat{\mathcal{F}}(s)$$

$$\begin{aligned}&= |\Omega(s)| e^{i\delta_1(s)} \left\{ \frac{C_1}{3} + \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\mathcal{F}(s') \sin \delta_1'(s')}{s'(s'-s) |\Omega(s')|} \right\} + \hat{\mathcal{F}}(s) \\ &= \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\mathcal{F}(s') \sin \delta_1'(s')}{s'(s'-s) |\Omega(s')|} + \frac{s}{\pi} i\pi \frac{\hat{\mathcal{F}}(s) \sin \delta_1'(s)}{s |\Omega(s)|} \\ &= |\Omega(s)| e^{i\delta_1(s)} \left\{ \frac{C_1}{3} + \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\mathcal{F}(s') \sin \delta_1'(s')}{s'(s'-s) |\Omega(s')|} \right\} \\ &\quad + e^{i\delta_1(s)} \left[ i\pi \hat{\mathcal{F}}(s) \sin \delta_1'(s) \right] + \hat{\mathcal{F}}(s) \\ &= \hat{\mathcal{F}}(s) \left[ 1 + i e^{i\delta_1(s)} \frac{\sin \delta_1'(s)}{\sin \delta_1(s)} \right] \\ &= 1 + \frac{e^{2i\delta_1(s)} - 1}{2} = \frac{1}{2} (1 + e^{2i\delta_1(s)}) = e^{i\delta_1(s)} \cos \delta_1(s) \\ &= e^{i\delta_1(s)} \left\{ \hat{\mathcal{F}}(s) \cos \delta_1(s) + |\Omega(s)| \left[ \frac{C_1}{3} + \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\mathcal{F}(s') \sin \delta_1'(s')}{s'(s'-s) |\Omega(s')|} \right] \right\},\end{aligned}$$

i.e. the representation fulfills Watson's final-state theorem (as expected, that is we only checked it explicitly); note that one can read off  $|f_1(s)|$  from this immediately. Similarly, one finds that for the second representation of  $f_1(s)$ , we have



$$f_1(s) = e^{i\delta_1^+(s)} \left\{ F(s) \cos \delta_1^+(s) + |\Omega(s)| \left[ \frac{C_2^{(1)}}{3} (1 - \Omega(s)) + \frac{C_2^{(1)}}{3} s + \frac{s^2}{4\pi^2} \int_{-\infty}^{\infty} ds' \frac{F(s') \sin \delta_1^+(s')}{s'^2 (s'-s) |\Omega(s')|} \right] \right\}$$

From this, one can then reconstruct the full amplitude  $F(s, t, u)$  by using (\*) and (\*\*\*) in combination with Watson's final-state theorem.