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Dispersive Analysis of the ρ Transition Form Factor 1

14.05.2021 We want to confirm some calculations from $\langle 1410.4691 \rangle$ (Hofmeister et al.); some (additional) calculations are done in a separate Mathematica NB. Furthermore, the $e^+e^- \rightarrow 3\pi$ cross section is calculated on a different sheet.

For the $\rho^+ \rightarrow 3\pi$ amplitude, we define

$$\langle 0 | j_\mu(0) | \pi^+(p_+) \pi^-(p_-) \pi^0(p_0) \rangle = -G_{\rho\pi\pi} p_+^\nu p_-^\alpha p_0^\beta F(s, t, u, q^2)$$

where $q = p_+ + p_- + p_0$, $s = (p_+ + p_-)^2$, $t = (p_- + p_0)^2$, $u = (p_+ + p_0)^2$, and $s + t + u = 3M_\pi^2 + q^2$.

Using that $s = (p_+ + p_-)^2 = (q - p_0)^2$, we find for the energies and momenta in the CMS of (ρ^+) and $(\pi^+ \pi^-)$ that

$$|\vec{p}_\pm| = \frac{\sqrt{\lambda(s, q^2, M_\pi^2)}}{2\sqrt{s}} = |\vec{p}_0| \quad \Rightarrow \quad \vec{p}_\pm = \vec{p}_0,$$

$$E_\pm = \frac{q^2 + s - M_\pi^2}{2\sqrt{s}}, \quad E_0 = \frac{q^2 - s - M_\pi^2}{2\sqrt{s}},$$

$$|\vec{p}_+| = \frac{\sqrt{s}}{2} \underbrace{\sigma_\pi(s)}_{= \sqrt{1 - \frac{4M_\pi^2}{s}}} = |\vec{p}_-| \quad \Rightarrow \quad \vec{p}_+ = -\vec{p}_-,$$

$$E_+ = \frac{\sqrt{s}}{2} = E_-.$$

Defining the s-channel scattering angle via

$$\vec{p}_+ \cdot \vec{p}_0 = |\vec{p}_+| |\vec{p}_0| \cos \theta_s, \quad \text{we have}$$

$$\vec{p}_- \cdot \vec{p}_0 = -\vec{p}_+ \cdot \vec{p}_0 = -|\vec{p}_+| |\vec{p}_0| \underbrace{\cos \theta_s}_{= -2},$$

so that

$$t = \frac{1}{2}(3M_\pi^2 + q^2 - s) + \frac{1}{2}k(s, q^2)\epsilon, \quad k(s, q^2) = \sigma_\pi(s) \sqrt{\lambda(s, q^2, M_\pi^2)}$$

An immediate consequence is

$$\cos \theta_s = \frac{t-u}{R(s, q^2)}$$

For the two-pion transition form factor, we define (see <1808.04823>, Hoyer et al.)

$$i \int d^4x e^{iq_1 \cdot x} \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | \pi^0(q_1+q_2) \rangle \\ = -\epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{\pi^0}^{\gamma\delta}(q_1^2, q_2^2)$$

We decompose the TFF into definite isospin components according to

$$F_{\pi^0}^{\gamma\delta}(q_1^2, q_2^2) = F_{\gamma\delta}^{\nu s}(q_1^2, q_2^2) + (q_1 \leftrightarrow q_2),$$

where the first (second) index refers to isovector (v) and isoscalar (s) quantum numbers of the photon with momenta q_1, q_2 . Using the two-pion discontinuity as calculated in <1206.3098> (Schneider et al.),

$$\text{disc } f_{\pi^0}^{\nu s}(s) = \frac{iq_\pi^3(s)}{6\pi\sqrt{s}} F_{\pi^0}^{\nu*}(s) f(s) \theta(s-4m_\pi^2), \quad q_\pi(s) = \sqrt{\frac{s}{4} - m_\pi^2}$$

we can write a once-subtracted dispersion relation in the isovector virtuality (for fixed isoscalar virtuality),

$$F_{\nu s}(s_1, s_2) = F_{\nu s}(0, s_2) + \frac{es_1}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_{\pi^0}^{\nu*}(s') f(s', s_2)}{s'^{3/2} (s'-s_1)}$$

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Assuming $F_{\pi^0}^{\nu*}(s)$ and $f(s, s_2)$ to asymptotically fall off like $1/s$ (for fixed s_2), there is a sum rule for the subtracted function $F_{\nu s}(0, s_2)$,

$$F_{\nu s}(s_1, s_2) = \frac{e}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_{\pi^0}^{\nu*}(s') f(s', s_2)}{s'^{3/2} (s'-s_1)}$$

$$\frac{1}{s'-s_1} = \frac{s_1}{s'(s'-s_1)} + \frac{1}{s'} \\ \frac{es_1}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_{\pi^0}^{\nu*}(s') f(s', s_2)}{s'^{3/2} (s'-s_1)} + \frac{e}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_{\pi^0}^{\nu*}(s') f(s', s_2)}{s'^{3/2}}$$

$$\Rightarrow F_{\nu s}(0, s_2) = \frac{e}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s')}{s'^{3/2}} F_{\pi^0}^{\nu*}(s') f(s', s_2)$$

Dispersive Analysis of the Pion Transition Form Factor 2

We can perform a (necessarily less explicit) subtraction in s_2 as well, defining a subtracted partial wave

$$\bar{f}_1(s, q^2) = \frac{f_1(s, q^2) - f_1(s, 0)}{q^2}$$

$$\Leftrightarrow f_1(s, q^2) = q^2 \bar{f}_1(s, q^2) + f_1(s, 0),$$

so that

$$\begin{aligned} F_{\pi S}(s_1, s_2) &= F_{\pi S}(0, s_2) + \frac{e s_1}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_\pi^{V^*}(s') f_1(s', s_2)}{s'^{3/2}(s'-s_1)} \\ &= F_{\pi S}(0, s_2) + \frac{e s_1 s_2}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_\pi^{V^*}(s') \bar{f}_1(s', s_2)}{s'^{3/2}(s'-s_1)} \\ &\quad + \frac{e s_1}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_\pi^{V^*}(s') f_1(s', 0)}{s'^{3/2}(s'-s_1)} \\ &\stackrel{(*)}{=} \frac{e}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_\pi^{V^*}(s') f_1(s', 0) [s_1 - s' + s']}{s'^{3/2}(s'-s_1)} \\ &= \underbrace{-\frac{e}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_\pi^{V^*}(s') f_1(s', 0)}{s'^{3/2}}}_{=-F_{\pi S}(0, 0)} + \underbrace{\frac{e}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_\pi^{V^*}(s') f_1(s', 0)}{s'^{3/2}(s'-s_1)}}_{=F_{\pi S}(s_1, 0)} \end{aligned}$$

or, alternatively,

$$(*) = F_{\pi S}(s_1, 0) - F_{\pi S}(0, 0)$$

$$= F_{\pi S}(s_1, 0) + F_{\pi S}(0, s_2) - \frac{F_{\pi S}(0, 0)}{2} + \frac{e s_1 s_2}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi^3(s') F_\pi^{V^*}(s') \bar{f}_1(s', s_2)}{s'^{3/2}(s'-s_1)}$$

(Note that one can also start from Eq. (31) and insert the sum rule as well as $F_{\pi S}(s_1, 0)$ to obtain the desired result.)

We now specialize to the singly-virtual case; the $n=0$ soft TFF can be written out explicitly according to

For the alternative, we do not have to use the sum rule? Why stated in paper that we have to "make use of sum rule again"?

$$F_{\text{reg}}(q^2, 0) = \underbrace{F_{\text{vs}}(q^2, 0)} + F_{\text{vs}}(0, q^2)$$

insert sum rule for $F_{\text{vs}}(0, q^2)$

$$= F_{\text{vs}}(0, 0) + \frac{e q^2}{12\pi^2} \int_{4m^2}^{\infty} ds' \frac{q_{\text{v}}^3(s') F_{\text{K}}^{\text{vK}}(s') f_1(s', 0)}{s'^{3/2} (s' - q^2)}$$

$$+ \frac{e}{12\pi^2} \int_{4m^2}^{\infty} ds' \frac{q_{\text{v}}^3(s')}{s'^{3/2}} F_{\text{K}}^{\text{vK}}(s') f_1(s', q^2)$$

$$+ \underbrace{F_{\text{vs}}(0, 0)} - \underbrace{F_{\text{vs}}(0, 0)}$$

$$= \frac{F_{\text{vs}}}{2} = - \frac{e}{12\pi^2} \int_{4m^2}^{\infty} ds' \frac{q_{\text{v}}^3(s')}{s'^{3/2}} F_{\text{K}}^{\text{vK}}(s') f_1(s', 0)$$

$$= \frac{F_{\text{vs}}}{2} + \frac{e}{12\pi^2} \int_{4m^2}^{\infty} ds' \frac{q_{\text{v}}^3(s') F_{\text{K}}^{\text{vK}}(s')}{s'^{3/2}} \left\{ \frac{q^2}{s' - q^2} f_1(s', 0) - f_1(s', 0) + f_1(s', q^2) \right\}$$

Why remove the TFF exactly like this? There are more/different possibilities!!

In order to find a relation between the pion transition form factor and the cross section $\sigma_{\text{etc} \rightarrow \pi\pi}$, we note that

$$= \text{Tr}(p_2) (-i\gamma_k) u_s(p_1) \frac{-i\gamma_k p_1}{q^2}$$

$$\times \left[-\epsilon_{\mu\nu\alpha\beta} p_0^\alpha p_3^\beta F_{\text{reg}}(q^2, p_3^2) \right] E^{\nu*}(p_1)$$

$\uparrow q = p_0 + p_3$ $\uparrow = 0$

So that the spin-averaged amplitude squared becomes

$$|M|^2 = \frac{1}{4} \text{Tr} \left\{ (\not{p}_2 - m_e) \gamma^\mu (\not{p}_1 + m_e) \gamma^\nu \right\} \epsilon_{\mu\nu\alpha\beta} p_0^\alpha p_3^\beta \epsilon_{\nu\alpha\beta\mu} p_0^\alpha p_3^\beta \epsilon_{\mu\nu\alpha\beta} p_0^\alpha p_3^\beta$$

$$\times (-g^{\mu\nu}) \left| F_{\text{reg}}(q^2, p_3^2) \right|^2 \frac{e^2}{q^4}$$

Inserting this into the cross section formula

$$\sigma_{\text{etc} \rightarrow \pi\pi} = \frac{1}{64\pi^2 q^2} \frac{|\vec{p}_{\text{out}}|}{|\vec{p}_{\text{in}}|} \int d\Omega |M|^2$$

Using Mathematica and assuming $m_e = 0$, we had

$$\sigma_{\text{etc} \rightarrow \pi\pi} = \frac{e^2 (q^2 - 4m^2)}{64\pi q^6} |F_{\text{reg}}(q^2, 0)|^2$$

Dispersive Analysis of the Pion Transition Form Factor 3

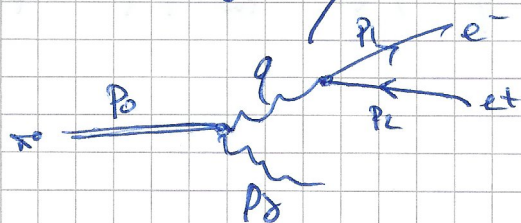
Last but not least, we want to calculate the decay width of the Dalitz decay $\pi^0 \rightarrow e^+e^-\gamma$, which we normalize to the two-photon decay. To this end, we first calculate the two-photon decay width $\pi^0 \rightarrow \gamma\gamma$,

$$|M_{\gamma\gamma}^2| = G_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta G^{\nu\lambda\mu\rho} q_1^\lambda q_2^\rho (-g^{\mu\nu}) (-g^{\lambda\rho}) \underbrace{|F_{\pi^0\gamma\gamma}(q,0)|^2}_{= F_{\pi^0\gamma\gamma}^2}$$

$$\Gamma_{\pi^0\gamma\gamma} = \frac{1}{2} \frac{|\vec{p}_\gamma|^3}{8\pi k^2} |M_{\gamma\gamma}^2| \quad (\text{see Master's Thesis})$$

$$= \frac{M_\pi^3}{64\pi} F_{\pi^0\gamma\gamma}^2.$$

For the decay $\pi^0 \rightarrow e^+e^-\gamma$, we calculate



$$= \bar{u}_s(p_1) (-ie\gamma_\mu) v_r(p_2) \frac{-ig^{\mu\nu}}{q^2}$$

$$\times (-G_{\mu\nu\alpha\beta} p_0^\alpha p_0^\beta \epsilon^{\nu\lambda\mu\rho} F_{\pi^0\gamma\gamma}(q,0))$$

\uparrow
 $q = p_0 - p_0$

$$\underbrace{m_{12}^2}_{=q^2} = (p_1 + p_2)^2 = 2m_e^2 + 2(p_1 \cdot p_2)$$

$$m_{23}^2 = (p_2 + p_\gamma)^2 = m_e^2 + 2(p_2 \cdot p_\gamma)$$

$$k^2 = p_0^2 = (p_1 + p_2 + p_\gamma)^2 = 2m_e^2 + 2(p_1 \cdot p_2) + 2(p_1 \cdot p_\gamma) + 2(p_2 \cdot p_\gamma).$$

The normalized ^{differential} decay width is then given by

$$\frac{d\Gamma_{\pi^0 \rightarrow e^+e^-\gamma}}{dq^2 dm_{23}^2 \Gamma_{\pi^0 \rightarrow \gamma\gamma}} = \frac{1}{\Gamma_{\pi^0 \rightarrow \gamma\gamma}} \frac{1}{(2\pi)^3} \frac{1}{32M_\pi^3} |M_{\pi^0 \rightarrow e^+e^-\gamma}|^2.$$

Integrating over m_{23}^2 , where the boundaries are given by

$$(M_{23}^2)_{\text{min}}^{\text{max}} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} \pm \sqrt{E_3^{*2} - m_3^2} \right)^2,$$

$$E_2^* = \frac{m_{12}^2 - m_1^2 + m_2^2}{2m_{12}} = \frac{m_{12}}{2}$$

$$E_3^* = \frac{M_0^2 - m_{12}^2 - m_3^2}{2m_{12}} = \frac{M_0^2 - m_{12}^2}{2m_{12}}$$

yields (Mathematik)

$$\begin{aligned} \frac{d\Gamma_{\mu \rightarrow e \nu \gamma}}{dq^2 \Gamma_{\mu \rightarrow e \nu \gamma}} &= \frac{e^2}{q^4} \left\{ \frac{q^2 \sqrt{q^2 - 4m_e^2} (2m_e^2 + q^2) \left(\frac{M_0^2 - q^2}{\sqrt{q^2}} \right)^3}{6\pi^2 F_{\text{box}}^2 M_0^6} \right\} \left| \frac{F_{\text{box}}(q^2, 0)}{F_{\text{box}}} \right|^2 \\ &= \frac{e^2}{6\pi^2} \frac{1}{q^2} \left\{ \frac{\sqrt{q^2} \sqrt{1 - \frac{4m_e^2}{q^2}}}{q^2} \frac{q^2 \left(1 + \frac{2m_e^2}{q^2} \right) \frac{M_0^6}{\sqrt{q^2}^3} \left(1 - \frac{q^2}{M_0^2} \right)^3}{M_0^6} \right\} \\ &\quad \times \left| \frac{F_{\text{box}}(q^2, 0)}{F_{\text{box}}} \right|^2 \\ &= \frac{e^2}{6\pi^2} \frac{1}{q^2} \sqrt{1 - \frac{4m_e^2}{q^2}} \left(1 + \frac{2m_e^2}{q^2} \right) \left(1 - \frac{q^2}{M_0^2} \right)^3 \left| \frac{F_{\text{box}}(q^2, 0)}{F_{\text{box}}} \right|^2 \end{aligned}$$