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Radiative Resonance Couplings in $\pi\pi \rightarrow \pi\pi$ 1

18.06.2021 We want to (re-)confirm some calculations from <1710.00824>; see also the corresponding Mathematica NB. Some calculations not done here explicitly might be found in the "Chiral Anomaly $\pi\pi \rightarrow \pi\pi$ " notes or NB as well as in my Bachelor's thesis.

$$\text{For } \mathcal{M}_{\pi\pi}^{I=1}(s,t) = 32\pi \sum_{\text{odd } l} (2l+1) t_l^1(s) P_l(z),$$

we note that in $\pi\pi \rightarrow \pi\pi$ scattering we have

$$|\vec{p}_\pi| = \frac{\sqrt{s-4m^2}}{2}, \quad E_\pi = \frac{\sqrt{s}}{2},$$

$$\text{so that } z = -\frac{1-z}{2} (s-4m^2)$$

$$\Rightarrow z = \frac{2t}{s-4m^2} + 1.$$

For the interaction Lagrangian of $\sigma\pi\pi$, we use an ansatz from Kling et al. (see also my Bachelor's Thesis),

$$\mathcal{L}_{\sigma\pi\pi} = \frac{ig}{4} \text{tr}(V^\mu [\partial_\mu \phi, \phi]), \quad \phi = \phi_a \tau_a \text{ in } \text{SU}(2)$$

$\rightarrow \phi = \phi_a \sigma_a \text{ in } \text{SU}(2)$,

so that in $\text{SU}(2)$

$$\begin{aligned} \mathcal{L}_{\sigma\pi\pi} &= \frac{ig}{4} \text{tr} \left\{ V^\mu (\partial_\mu \phi_i) \phi_j \overbrace{[\sigma_i, \sigma_j]}^{= 2i \epsilon_{ijk} \sigma_k} \right\} \\ &= \frac{-g}{2} \epsilon_{ijk} v_k^\mu (\partial_\mu \phi_i) \phi_j \underbrace{\text{tr} \{ \sigma_e \sigma_k \}}_{= 2\delta_{ek}} \end{aligned}$$

$$= -g \epsilon_{ijk} v_k^\mu (\partial_\mu \phi_i) \phi_j$$

$$= g \epsilon_{ijk} \phi_i (\partial_\mu \phi_j) v_k^\mu \approx g_{\text{SPT}} e^{\frac{abc}{\pi} a} \partial_\mu^b c e^\mu.$$

Note that transforming the fields ϕ and V into the charge basis, e.g.

$$\phi = \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & \frac{1}{\sqrt{3}}\phi_8 - \phi_3 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

see
Mathematica \rightarrow
$$\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}\pi^- & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}\pi^0 \\ \sqrt{2}\pi^+ & \sqrt{2}\pi^0 & -\frac{2}{\sqrt{3}}\eta_8 \end{pmatrix}$$

We find agreement with Khngl et al. for the matrices ϕ and V .

Transforming the Lagrangian into the charge basis and evaluating the expression with Mathematica, we find

$$\mathcal{L}_{\text{SM}} = ig_{\text{SM}} \left\{ \begin{aligned} & S^{0\mu} [\pi^- \partial_\mu \pi^+ - \pi^+ \partial_\mu \pi^-] \\ & + S^{+T} [\pi^0 \partial_\mu \pi^- - \pi^- \partial_\mu \pi^0] \\ & + S^{+T} [\pi^+ \partial_\mu \pi^0 - \pi^0 \partial_\mu \pi^+] \end{aligned} \right\},$$

so that

$$\mathcal{M}(S^0 \rightarrow \pi^+ \pi^-) = g_{\text{SM}} E_\mu(\mathbb{P}) (p_+ - p_-)^\mu,$$

$$\mathcal{M}(S^+ \rightarrow \pi^+ \pi^0) = g_{\text{SM}} E_\mu(\mathbb{P}) (p_+ - p_0)^\mu,$$

$$\mathcal{M}(S^- \rightarrow \pi^- \pi^0) = g_{\text{SM}} E_\mu(\mathbb{P}) (p_0 - p_-)^\mu,$$

where we dropped physical overall phase factors.

Radiative Resonance Couplings in $\pi^+ \rightarrow \pi^0 \gamma$

For the interaction Lagrangian of S^0 , we ^{can} use another ansatz from Kluge et al.,

$$\mathcal{L}_{S^0}^{(3)} = \frac{ic}{2f_\pi^3} \epsilon^{\mu\nu\alpha\beta} A_\mu \text{tr} (Q \partial_\nu \phi \partial_\alpha \phi \partial_\beta \phi) + \frac{d}{f_\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} \text{tr} (Q V_\alpha \partial_\beta \phi),$$

where we only need the ^{second} part; in $SU(2)$, this becomes

$$\begin{aligned} \mathcal{L}_{S^0}^{(3)} &= \frac{d}{f_\pi} \epsilon^{\mu\nu\alpha\beta} (\partial_\mu A_\nu - \partial_\nu A_\mu) V_\alpha \partial_\beta \phi_j \text{tr} (Q \sigma_i \sigma_j) \\ &\quad \left| \begin{aligned} \text{tr} (Q \sigma_i \sigma_j) &= \text{tr} (Q [\delta_{ij} \sigma_0 + i \epsilon_{ijk} \sigma_k]) \\ &= \frac{1}{3} \delta_{ij} + i \epsilon_{ijk} \delta_{k3} \end{aligned} \right. \\ &= \frac{2d}{f_\pi} \epsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu V_\alpha \partial_\beta \phi_j \left(\frac{1}{3} \delta_{ij} + i \epsilon_{ij3} \right). \end{aligned}$$

Transforming into the charge basis and evaluating the expression with Mathematica, we had

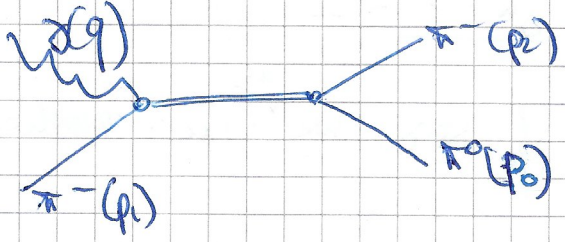
$$\begin{aligned} \mathcal{L}_{S^0} &= \frac{2d}{3f_\pi} \epsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu (S^0_\alpha \partial_\beta \pi^0 + 4S^+_\alpha \partial_\beta \pi^- - 2S^-_\alpha \partial_\beta \pi^+) \\ &\approx e g_{S^0} \epsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu (S^0_\alpha \partial_\beta \pi^0 + 4S^+_\alpha \partial_\beta \pi^- - 2S^-_\alpha \partial_\beta \pi^+) \end{aligned}$$

Why should the S^+ couple with twice the strength of the S^- ? Also, a factor $1/3$ ($1/3$ for squared amplitude) too much as compared to decay width of Kluge et al. and furthermore, for the ω coupling we had a factor $1/3$ too much. Also, need S^- ?

We now restrict to the matrix element

$$\begin{aligned} \mathcal{M}_{S^0} &= e g_{S^0} \epsilon^{\mu\nu\alpha\beta} p_2^\mu \epsilon_\nu^\alpha \epsilon_S^\beta p_1^\alpha p_2^\beta \\ &= e g_{S^0} \epsilon^{\mu\nu\alpha\beta} \epsilon_S^\mu \epsilon_\nu^\alpha p_1^\alpha p_2^\beta \end{aligned}$$

for the process $\pi^+ \rightarrow \pi^0 \gamma$ modelled by a S resonance in the VMD picture, we show that



$$\begin{aligned}
 &= \frac{e g_{\pi\pi\gamma} E_{\pi^0}^{\nu} E_{\pi^+}^{\mu} E_{\pi^-}^{\lambda} p_i^{\alpha} q^{\beta}}{i \left(g^{\mu\kappa} - \frac{p_3^{\mu} p_3^{\kappa}}{M_{\pi}^2} \right)} \\
 &= \frac{i e g_{\pi\pi\gamma} g_{\pi\pi\pi}}{s - M_{\pi}^2 + i M_{\pi} \Gamma_{\pi}} g_{\pi\pi\pi} (p_0 - p_2)_k \\
 &= \frac{i e g_{\pi\pi\gamma} g_{\pi\pi\pi}}{s - M_{\pi}^2 + i M_{\pi} \Gamma_{\pi}} E_{\pi^0}^{\nu} p_i^{\alpha} (p_2 + p_0 - p_1)^{\beta} (p_0 - p_2)^{\mu} \\
 &= -i E_{\pi^0}^{\nu} E_{\pi^+}^{\mu} p_i^{\alpha} (p_2 + p_0)^{\beta} (p_0 - p_2)^{\nu} \left[\frac{e g_{\pi\pi\gamma} g_{\pi\pi\pi}}{s - M_{\pi}^2 + i M_{\pi} \Gamma_{\pi}} \right] \\
 &= -i E_{\pi^0}^{\nu} E_{\pi^+}^{\mu} p_i^{\alpha} (p_2 + p_0)^{\beta} (p_0 - p_2)^{\nu} \left[\frac{e g_{\pi\pi\gamma} g_{\pi\pi\pi}}{s - M_{\pi}^2 + i M_{\pi} \Gamma_{\pi}} \right] \\
 &= i E_{\pi^0}^{\nu} E_{\pi^+}^{\mu} p_i^{\alpha} p_2^{\beta} p_0^{\nu} \left[\frac{2 e g_{\pi\pi\gamma} g_{\pi\pi\pi}}{M_{\pi}^2 - s - i M_{\pi} \Gamma_{\pi}} \right] \\
 &= \mathcal{F}(s, t, u)
 \end{aligned}$$

$$\Rightarrow f_{\pi}^{\text{MD}}(s) = \frac{2 e g_{\pi\pi\gamma} g_{\pi\pi\pi}}{M_{\pi}^2 - i M_{\pi} \Gamma_{\pi} - s}$$

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For the radiative width, we calculate the spin-averaged matrix element squared of $\mathcal{M}_{\pi\pi\gamma}$, using (Mathematica)

$$|\mathcal{M}_{\pi\pi\gamma}|^2 = \frac{e^2 |g_{\pi\pi\gamma}|^2}{6} (M_{\pi}^2 - M_{\pi}^2)^2$$

the decay width then becomes

$$\Gamma_{\pi \rightarrow \pi\gamma} = \frac{|\overline{p_{\text{out}}}|}{8\pi M_{\pi}^3} |\mathcal{M}_{\pi\pi\gamma}|^2 = \frac{e^2 |g_{\pi\pi\gamma}|^2}{96\pi M_{\pi}^3} (M_{\pi}^2 - M_{\pi}^2)^3$$

For the radiative width of the ρ_3 , $\Gamma_{\rho_3 \rightarrow \pi\gamma}$, we combine the amplitudes for $\mathcal{M}(\rho_3 \rightarrow \pi\omega)$ and $\mathcal{M}(\omega \rightarrow \gamma)$,

$$\mathcal{M}(\rho_3 \rightarrow \pi\omega) = \frac{g_{\rho\omega\pi}}{f_{\pi}} e^{\lambda\alpha\beta\gamma} E_{\rho_3\mu\nu} p_{\pi}^{\mu} p_{\pi}^{\nu} p_{\omega}^{\lambda} p_{\omega}^{\gamma} E_{\omega\delta}$$

$$\mathcal{M}(\omega \rightarrow \gamma) = -\frac{e M_{\omega}^2}{g_{\omega\gamma}} E_{\gamma}^{\mu} E_{\omega\nu}$$

today

Radiative Resonance Couplings in $\rho \rightarrow \pi\pi$

$$\begin{aligned}
 M(s_3 \rightarrow \pi\pi) &= \frac{e g_{\rho\pi\pi} M_\rho}{g_\omega F_\pi} e^{i\alpha_\rho} \epsilon_{\rho\pi\pi} \epsilon_{\rho\pi\pi}^T \epsilon_\rho^\nu \epsilon_\rho^\nu \epsilon_\rho^\nu \\
 &\times \frac{(-g_{\rho\pi\pi} + \frac{P_\rho^\mu P_{\rho\pi}^\mu}{M_\rho^2})}{P_\rho^2 - M_\rho^2} \epsilon_\rho^\mu \\
 &= \frac{e g_{\rho\pi\pi}}{g_\omega F_\pi} e^{i\alpha_\rho} \epsilon_{\rho\pi\pi} \epsilon_{\rho\pi\pi}^T \epsilon_\rho^\nu \epsilon_\rho^\nu \epsilon_\rho^\nu \epsilon_\rho^\mu
 \end{aligned}$$

The squared spin-averaged amplitude then becomes (see Mathematics; $\lambda = \eta$ for the index to avoid conflicts with Källén function and Bell-Man matrices)

$$|M(s_3 \rightarrow \pi\pi)|^2 = \frac{e^2 |g_{\rho\pi\pi}|^2}{8\pi F_\pi^2 |g_\omega|^2} \frac{(M_\rho^2 - M_{S_3}^2)^6}{M_{S_3}^4}$$

So that the decay width reads

$$\begin{aligned}
 \Gamma(s_3 \rightarrow \pi\pi) &= \frac{|\text{Phase}|}{8\pi M_{S_3}^2} |M(s_3 \rightarrow \pi\pi)|^2 \\
 &= \frac{e^2 |g_{\rho\pi\pi}|^2}{13440\pi F_\pi^2 |g_\omega|^2 M_{S_3}^7} (M_{S_3}^2 - M_\rho^2)^7
 \end{aligned}$$

In order to extract the $\rho(770)$ properties in a model-independent way, we now consider $I=1$ $\pi\pi$ scattering, which encodes the said properties as pole position and residues of the S-matrix on the second Riemann sheet. In the vicinity of the pole, the partial-wave amplitude may be written as

$$t_{1,1}^I(s) = \frac{g_{\rho\pi\pi}^2 (s - 4m_\pi^2)}{48\pi (s_3 - s)}, \quad s_3 = \left(M_\rho - i \frac{\Gamma_\rho}{2}\right)^2$$

(García-Martín et al.)

See also <1107.1635>. While it's clear that from a pole of first degree, we would expect the residue to fulfill

$$g_{\rho\pi\pi}^2 = \lim_{s \rightarrow s_3} (s - s_3) t_{1,1}^I(s),$$

The other kinematic factors are chosen such that in the narrow-width limit, the coupling g_{SM} matches onto the Lagrangian definition

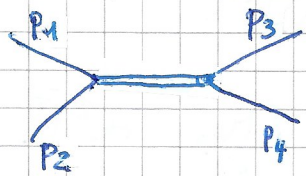
$$\mathcal{L}_{SM} = i g_{SM} \delta_F (\bar{n} \not{\partial} n^+ - n^+ \not{\partial} \bar{n})$$

and the real component of the coupling g_{SM} remains larger (positive) than the imaginary component (if we chose a different sign in $g_{SM} = \sqrt{\text{Re} g_{SM}^2 + i \text{Im} g_{SM}^2}$, we would find $\tilde{g}_{SM} = i g_{SM} = i (\text{Re} g_{SM} + i \text{Im} g_{SM}) = -\text{Im} g_{SM} + i \text{Re} g_{SM}$; see also <2009.04479> (Nehus et al.) for numerical values of the real and imaginary part). More specifically, the Lagrangian above leads to

see my Beeder's Thesis for factor of i (without the minus sign below, would end up with a missing sign, which could, however, also be fixed by hand $-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_S^2}$)

$$\mathcal{M}(g^0 \rightarrow \pi^+ \pi^-) = i g_{SM} \text{Esp} (p_2 - p_1)^\mu$$

So that



$$= -g_{SM}^2 (p_4 - p_3)_\mu (p_2 - p_1)^\mu \frac{1}{p_S^2 - M_S^2 + i M_S \Gamma_S}$$

$$= g_{SM}^2 (p_2 - p_1) \cdot (p_4 - p_3) \frac{1}{p_S^2 - M_S^2 + i M_S \Gamma_S}$$

$$= -g_{SM}^2 ((p_1 \cdot p_4) + (p_1 \cdot p_3) - (p_2 \cdot p_4) + (p_2 \cdot p_3)) \frac{1}{p_S^2 - M_S^2 + i M_S \Gamma_S}$$

probably also factor of i for the propagator! But only if Feynman rules for $i\mathcal{M}$!!

Working out the kinematics in Mathematica (i.e. using an explicit frame of reference with angle $z = \cos \theta$), we find

$$\mathcal{M}(n\bar{n} \rightarrow S \rightarrow n\bar{n}) = -g_{SM}^2 z (s - 4M_S^2) \frac{1}{p_S^2 - M_S^2 + i M_S \Gamma_S}$$

Using the orthogonality relation of the Legendre polynomials,

$$\int_{-1}^1 dz P_n(z) P_m(z) = \frac{2}{2n+1} \delta_{n,m},$$

We can furthermore invert the partial-wave decomposition

Radiative Resonance Couplings in $\pi\pi \rightarrow \pi\pi$ 4

$$M_{\pi\pi}^{I=1}(s,t) = 16\pi \sum_{\text{odd } l} (2l+1) t_l^1(s) P_l(z)$$

(note that we replaced 32π by 16π)

according to

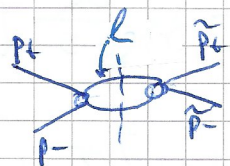
$$\begin{aligned} \int_{-1}^1 dz M_{\pi\pi}^{I=1}(s,t) P_l(z) &= \int_{-1}^1 dz P_l(z) \left\{ 16\pi \sum_{\text{odd } l'} (2l'+1) t_{l'}^1(s) P_{l'}(z) \right\} \\ &= 16\pi \sum_{\text{odd } l'} (2l'+1) t_{l'}^1(s) \underbrace{\int_{-1}^1 dz P_l(z) P_{l'}(z)}_{= \frac{2}{2l+1} \delta_{ll'}} \\ &= 32\pi t_l^1(s), \end{aligned}$$

so that we find

$$\begin{aligned} t_l^1(s) &= \frac{1}{32\pi} \int_{-1}^1 dz M(\pi\pi \rightarrow \pi\pi) P_l(z) \\ &= -\frac{g_{\pi\pi}^2 (s-4m_\pi^2)}{48\pi (s-M_\rho^2)} = \frac{g_{\pi\pi}^2 (s-4m_\pi^2)}{48\pi (M_\rho^2-s)} \end{aligned}$$

If we did not have the additional sign in the beginning, would end up with wrong sign here? But invariant amplitude only fixed up to an unobservable phase?

in the narrow-width limit.

For the discontinuity of $\pi\pi$ scattering, , we use

$$M_{\pi\pi}^{I=1}(s, \epsilon) = 32\pi \sum_{\text{odd } l} (2l+1) t_l^1(s) P_l(z)$$

and calculate

$$\begin{aligned} &\frac{1}{2} \text{disc} \left\{ 32\pi \sum_{\text{odd } l} (2l+1) t_l^1(s) P_l(z) \right\} \\ &= \frac{i}{4} \int \frac{d^4 l}{(2\pi)^4} M_{\pi\pi}^{I=1}(s, z) M_{\pi\pi}^{I=1}(s, z^*) \langle \pi | \delta(l^2 - m_\pi^2) | \pi \rangle \delta((q-l)^2 - m_\pi^2) \\ &= \frac{i}{16\pi^2} \int d^4 l \left\{ 32\pi \sum_{\text{odd } l'} (2l'+1) t_{l'}^1(s) P_{l'}(z') \right\} \left\{ 32\pi \sum_{\text{odd } l''} (2l''+1) t_{l''}^{1*}(s) P_{l''}(z'') \right\} \\ &\quad \times \delta(l^2 - m_\pi^2) \delta((q-l)^2 - m_\pi^2) \end{aligned}$$

$q = \vec{p}_1 + \vec{p}_2$

$$\bullet q = \sqrt{p_+^2 + p_-^2} \Rightarrow q^2 = s$$

$$\bullet q^0 = \sqrt{s} \text{ w. CRCS}$$

$$\bullet \delta(l^2 - h_0^2) = \delta((l^0)^2 - (\vec{l}^2 + h_0^2))$$

$$\delta(l^2 - h_0^2) = \delta(q^2 + l^2 - 2q \cdot l - h_0^2)$$

$$\bullet \delta(f(x)) = \sum_{x_0} \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

$$\bullet d^4l = dl^0 d\vec{l} d\Omega_{\vec{l}} dl^2$$

$$\bullet l^0 = \sqrt{\vec{l}^2 + h_0^2} \Rightarrow \frac{dl^0}{d\vec{l}} = \frac{\vec{l}}{l^0} \Rightarrow dl^0 l^0 = d\vec{l} \cdot \vec{l}$$

$$= 64i \int dl^0 d\vec{l} d\Omega_{\vec{l}} dl^2 \left\{ \sum_{\text{odd } l^1} (2l^1 + 1) t_{l^1}^1(s) P_{l^1}(z^1) \right\} \\ \times \left\{ \sum_{\text{odd } l^2} (2l^2 + 1) t_{l^2}^1(s)^* P_{l^2}(z^2) \right\} \delta((l^0)^2 - (\vec{l}^2 + h_0^2)) \delta(s - 2q^0 \cdot l^0)$$

$$= 64i \int \frac{d\vec{l} d\Omega_{\vec{l}} dl^2}{2l^0} |\vec{l}|^2 \left\{ \sum_{\text{odd } l^1} (2l^1 + 1) t_{l^1}^1(s) P_{l^1}(z^1) \right\} \left\{ \sum_{\text{odd } l^2} (2l^2 + 1) t_{l^2}^1(s)^* P_{l^2}(z^2) \right\} \\ \times \delta(s - 2\sqrt{s} l^0)$$

$$= \delta(2\sqrt{s}(l^0 - \frac{s}{2\sqrt{s}}))$$

$$|\vec{l}| = \sqrt{\frac{s}{4} - h_0^2}$$

$$\stackrel{\int}{=} 64i \int \frac{dl^0 d\Omega_{\vec{l}} dl^2}{2} \sqrt{\frac{s}{4} - h_0^2} \left\{ \sum_{\text{odd } l^1} (2l^1 + 1) t_{l^1}^1(s) P_{l^1}(z^1) \right\} \left\{ \sum_{\text{odd } l^2} (2l^2 + 1) t_{l^2}^1(s)^* P_{l^2}(z^2) \right\} \\ \times \delta(2\sqrt{s}(l^0 - \frac{s}{2\sqrt{s}}))$$

$$= \frac{16i}{\sqrt{s}} \int d\Omega_{\vec{l}} \frac{\sqrt{s}}{2} \sqrt{1 - \frac{4h_0^2}{s}} \left\{ \sum_{\text{odd } l^1} (2l^1 + 1) t_{l^1}^1(s) P_{l^1}(z^1) \right\} \left\{ \sum_{\text{odd } l^2} (2l^2 + 1) t_{l^2}^1(s)^* P_{l^2}(z^2) \right\}$$

$$= 8i \sqrt{1 - \frac{4h_0^2}{s}} \int d\Omega_{\vec{l}} \left\{ \sum_{\text{odd } l^1} (2l^1 + 1) t_{l^1}^1(s) P_{l^1}(z^1) \right\} \left\{ \sum_{\text{odd } l^2} (2l^2 + 1) t_{l^2}^1(s)^* P_{l^2}(z^2) \right\}$$

$$\int d\Omega_{\vec{l}} P_l(z^1) P_{l'}(z^2) = \frac{4\pi}{2l+1} \delta_{l,l'} P_l(z)$$

(see Dissertation Schneider)

$$= 32\pi i \sqrt{1 - \frac{4h_0^2}{s}} \sum_{\text{odd } l} (2l+1) t_l^1(s) t_l^1(s)^* P_l(z)$$

$$\Rightarrow \text{disc } t_l^1(s) = 2i \sqrt{1 - \frac{4h_0^2}{s}} t_l^1(s) t_l^1(s)^*$$

Radiative Resonance Couplings in $\mathcal{PT} \rightarrow \mathcal{PT}^2$

24.06.2021

This discontinuity equation for $t_l^{\pm}(s)$ (unitarity relation) is valid in the physical region, i.e. along the positive real axis on the first Riemann sheet from the onset of the discontinuity onwards when approaching the cut from above (on the real axis below the threshold the discontinuity vanishes). Note that from general principles of QFT (\mathcal{T} -inv. see Heussallan, 2011, <1110.6074>), the amplitude on the first Riemann sheet fulfills Schwarz' reflection principle, $t_l^{\pm}(s^*) = t_l^{\pm}(s)^*$.

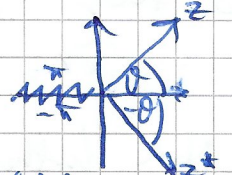
The discontinuity equation can then be written as (restricting to $l=1$ from now on) for $s \geq 4m^2$, $\sigma_{\pi}(s_{\pm}) = \sigma_{\pi}(s)$ with the principal square root

$$t_1^+(s+i\epsilon) - t_1^+(s-i\epsilon) = 2i \sigma_{\pi}(s+i\epsilon) \underbrace{t_1^+(s+i\epsilon) t_1^-(s+i\epsilon)}^*.$$

However, the function $\sigma_{\pi}(s+i\epsilon)$ has a cut (principal square root) from $s \in (0, 4m^2)$, whereas we need it to have cuts from $s \in (-\infty, 0)$ and $s \in (4m^2, \infty)$ in order to have a consistently chosen cut-structure of the equation above.

To see this, note that $\sigma_{\pi}(s) = \sqrt{1 - \frac{4m^2}{s}}$ has a negative argument for $s \in (0, 4m^2)$, so that $\sigma_{\pi}(s)$ destroys the structure of the partial wave on the second sheet; in fact, we will see that $(i\sigma_{\pi}(s))$ needs to fulfill Schwarz' reflection principle in order for the amplitude on the second sheet to fulfill Schwarz' reflection principle, while $\sigma_{\pi}(s)$ fulfills Schwarz' reflection principle itself (principal square root).

$$\sigma_{\pi}(s_{\pm}) = \sqrt{1 - \frac{4m^2}{s_{\pm}}} = \sqrt{1 - \frac{4m^2}{s} \pm i\epsilon}$$



$$\Rightarrow \sqrt{z^*} = \sqrt{|z|} \exp(i \frac{\arg(z^*)}{2}) = \sqrt{|z|} \exp(-i \frac{\arg(z)}{2}) = \sqrt{z^*}$$

so that

$$(i\sigma_{\pi}(s))^* = -i\sigma_{\pi}^*(s) = -i\sigma_{\pi}(s^*) = -(i\sigma_{\pi}(s^*))$$

i.e. $(i\sigma_{\pi}(s))$ does not fulfil Schwarz' reflection principle in this case. As a general principle, one has to think about the correct cut-structure of the discontinuity equation

$$t_1^+(s_+) - t_1^-(s_-) = 2i\sigma_{\pi}(s_+) t_1^+(s_+) \underbrace{t_1^-(s_-)^*}_{= \sigma_{\pi}(s) \text{ with principal square root}}$$

As soon as one goes away from the physical region $s=s_+$ with $\epsilon \rightarrow 0$. A cure for this problem can be achieved via two different methods: either by choosing the branch cut of the square root from $z \in (0, \infty)$ instead of $z \in (-\infty, 0)$,

$$\sqrt{z} = \begin{cases} \sqrt{z} & ; \operatorname{Im} z \geq 0 \\ -\sqrt{z} & ; \operatorname{Im} z < 0 \end{cases}$$

or by defining a different phase-space "factor",

$$\sigma_{\pi}(s) = \sqrt{\frac{4k_0^2}{s} - 1}$$

Using the first method, we have

$$\tilde{\sigma}_{\pi}(s_{\pm}) = \sqrt{1 - \frac{4k_0^2}{s_{\pm}}} = \sqrt{1 - \frac{4k_0^2}{s} \pm i\epsilon} = \begin{cases} \sqrt{1 - \frac{4k_0^2}{s} + i\epsilon} & ; s=s_+ \\ -\sqrt{1 - \frac{4k_0^2}{s} - i\epsilon} & ; s=s_- \end{cases}$$

≥ 0 for $s \in (-\infty, 0)$
and $s \in (4k_0^2, \infty)$

$$\begin{aligned} \sqrt{z^*} &= \begin{cases} -\sqrt{z^*} & ; \operatorname{Im} z \geq 0 \\ \sqrt{z^*} & ; \operatorname{Im} z < 0 \end{cases} = \begin{cases} -\sqrt{z}^* & ; \operatorname{Im} z \geq 0 \\ \sqrt{z}^* & ; \operatorname{Im} z < 0 \end{cases} = \begin{cases} -\sqrt{z}^* \\ -\sqrt{z}^* \end{cases} \\ &= -\sqrt{z}^* \end{aligned}$$

so that

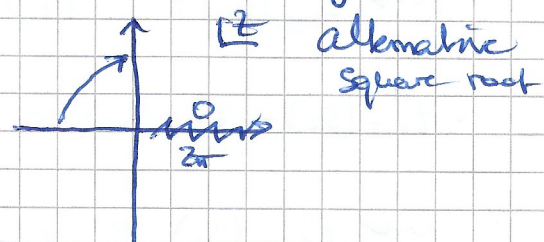
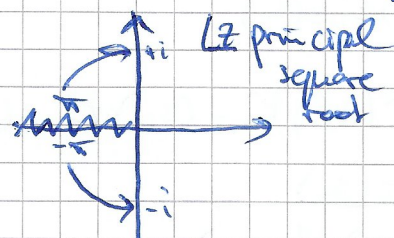
$$i\sqrt{z^*} = -i\sqrt{z}^* = (i\sqrt{z})^*$$

Note furthermore that for the principal square root, we have $\sqrt{-1} = \pm i$, where one has to decide for one of the values,

Radiative Resonance Couplings in $\pi\pi \rightarrow \pi\pi$ 6

while with the alternative square root, one always has

$$\sqrt{-1} = +i :$$



For the other method, we now investigate

$$\begin{aligned} \sigma^*(s \pm i\epsilon) &= \sqrt{\frac{4M_{\pi}^2}{s \pm i\epsilon} - 1} = \sqrt{\frac{4M_{\pi}^2}{s} - 1 \mp i\epsilon} \\ &\leq 0 \text{ for } s \in (-\infty, 0) \\ &\text{and } s \in (4M_{\pi}^2, \infty) \end{aligned}$$

$$= \sqrt{|z_{\pm}|} e^{i \arg(z_{\pm})/2} = \sqrt{|z_{\pm}|} e^{\mp i\pi/2}$$

$$= \mp i \sqrt{|z_{\pm}|} = \mp i \sqrt{1 - \frac{4M_{\pi}^2}{s}} = \mp i \sigma_{\pi}(s),$$

where, in particular, $\sigma^*(s^*) = \sigma^*(s)^*$, already from the considerations of the principal square root before.

We now rewrite the discontinuity equation according to

$$t_{\text{I}}^+(s+i\epsilon) - t_{\text{I}}^+(s-i\epsilon) = \mp 2\sigma^*(s \pm i\epsilon) t_{\text{I}}^+(s+i\epsilon) \underbrace{t_{\text{I}}^+(s+i\epsilon)^*}_{= t_{\text{I}}^+(s-i\epsilon)}$$

One then demands continuity to the second sheet, generically in the form $t_{\text{I}}(s_+) = t_{\text{II}}(s_-)$, which is the correct/canonical choice for more than two sheets; if there are only two sheets / all phase spaces (for each channel) are square roots, then the choice $t_{\text{I}}(s_-) = t_{\text{II}}(s_+)$ works equally well, which, however, is problematic for more than two sheets because $t_{\text{I}}(s_-)$ in the limit $\epsilon \rightarrow 0$ might bring one to another sheet (note that the label "2" can be put on any sheet for more than two sheets). Using either of these continuity relations, we then find

$$\bullet t_{1,II}^1(s_-) - t_{1,II}^1(s_-) = \mp 2\sigma^*(s_+) t_{1,II}^1(s_-) t_{1,II}^1(s_-)$$

$$\leadsto t_{1,II}^1(s_-) = \frac{t_{1,II}^1(s_-)}{1 \pm 2\sigma^*(s_+) t_{1,II}^1(s_-)}$$

$$\bullet t_{1,II}^1(s_+) - t_{1,II}^1(s_+) = \mp 2\sigma^*(s_+) t_{1,II}^1(s_+) t_{1,II}^1(s_+)$$

$$\leadsto t_{1,II}^1(s_+) = \frac{t_{1,II}^1(s_+)}{1 \mp 2\sigma^*(s_+) t_{1,II}^1(s_+)}$$

and the proper choice of s_+ in $\sigma^*(s_+)$ gives equivalent results.

Moreover, we have

$$\begin{aligned} t_{1,II}^1(s_-)^* &= \frac{t_{1,II}^1(s_-)^*}{1 - 2\sigma^*(s_-)^* t_{1,II}^1(s_-)^*} = \frac{t_{1,II}^1(s_+)}{1 - 2\sigma^*(s_+) t_{1,II}^1(s_+)} \\ &= t_{1,II}^1(s_+). \end{aligned}$$

Note that none of this is in contradiction with the following observations, starting from the discontinuity equation with $\sigma_{\mathbb{R}}(s)$:

$$\left| \begin{aligned} t_1^1(s_+) - t_1^1(s_-) &= 2i\sigma_{\mathbb{R}}(s) t_1^1(s_+) \underbrace{t_1^1(s_+)^*}_{= t_1^1(s_-)} = 2i\sigma_{\mathbb{R}}(s) t_1^1(s_-)^* t_1^1(s_-) \\ &\quad \swarrow \text{Schwarz' reflection principle} \end{aligned} \right.$$

$$\bullet t_{1,II}^1(s_-) - t_{1,II}^1(s_-) = 2i\sigma_{\mathbb{R}}(s) t_{1,II}^1(s_-) t_{1,II}^1(s_-)$$

$$\leadsto t_{1,II}^1(s_-) = \frac{t_{1,II}^1(s_-)}{1 - 2i\sigma_{\mathbb{R}}(s) t_{1,II}^1(s_-)}$$

$$\bullet t_{1,II}^1(s_+) - t_{1,II}^1(s_+) = 2i\sigma_{\mathbb{R}}(s) t_{1,II}^1(s_+) t_{1,II}^1(s_+)$$

$$\leadsto t_{1,II}^1(s_+) = \frac{t_{1,II}^1(s_+)}{1 + 2i\sigma_{\mathbb{R}}(s) t_{1,II}^1(s_+)},$$

in particular,

$$t_{1,II}^1(s_-)^* = \frac{t_{1,II}^1(s_-)^*}{1 + 2i\sigma_{\mathbb{R}}^*(s) t_{1,II}^1(s_-)^*} = \frac{t_{1,II}^1(s_+)}{1 + 2i\sigma_{\mathbb{R}}(s^*) t_{1,II}^1(s_+)} \neq t_{1,II}^1(s_+)$$

for the first relation (but correct for the combination of the first and second relation), merely because we haven't chosen the branch-cut structure $\sigma_{\mathbb{R}}(s)$ consistently here

Radiative Resonance Coupling in $\pi \rightarrow \pi \pi$

The pole parameters can then readily be determined from the condition

$$s_0 = \left(M_\rho - i \frac{\Gamma_\rho}{2} \right)^2, \text{ i.e. } \text{Im} s_0 < 0$$

$$t_{1,\pi}^1(s_0) = \frac{1}{20^\pi(s_0)} = \frac{1}{2i0^\pi(s_0)} = \frac{-i}{20^\pi(s_0)},$$

Once a ^{reliable} representation of $t_1^1(s)$ on the first sheet is available.

In analogy to what we did before, the elastic unitarity relation for the pion vector form factor,

$$\text{Im} F_\pi^V(s) = 0^\pi(s) \left(t_1^1(s) \right)^* F_\pi^V(s) \theta(s - 4m_\pi^2)$$

$$\Leftrightarrow F_\pi^V(s_+) - F_\pi^V(s_-) = 2i0^\pi(s) \left(t_1^1(s) \right)^* F_\pi^V(s)$$

defines the analytic continuation of the form factor onto the second sheet as per ^(*) Note that this is equivalent to the relation in the paper:

$$F_{\pi,\pi}^V(s) - F_{\pi,\pi}^V(s_-) = -20^\pi(s) F_{\pi,\pi}^V(s) t_{1,\pi}^1(s),$$

for $s = s_+$ above the cut

$$\begin{aligned} \bullet F_{\pi,\pi}^V(s_-) - F_{\pi,\pi}^V(s_+) &= \mp 20^\pi(s_\pm) \left(t_{1,\pi}^1(s_\pm) \right)^* F_\pi^V(s_\pm) \\ &= t_{1,\pi}^1(s_-) = F_{\pi,\pi}^V(s_-) \end{aligned}$$

$$\Rightarrow F_{\pi,\pi}^V(s_-) = \frac{F_{\pi,\pi}^V(s_+)}{1 \pm 20^\pi(s_\pm) t_{1,\pi}^1(s_\pm)}$$

$$\bullet F_{\pi,\pi}^V(s_+) - F_{\pi,\pi}^V(s_-) \stackrel{(*)}{=} \mp 20^\pi(s_\pm) \frac{t_{1,\pi}^1(s_\pm) F_{\pi,\pi}^V(s_+)}{\left(t_{1,\pi}^1(s_\pm) \right)^*}$$

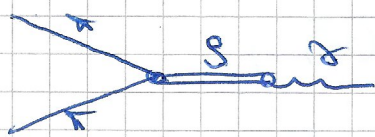
$$\Rightarrow F_{\pi,\pi}^V(s_+) = \frac{F_{\pi,\pi}^V(s_-)}{1 \pm 20^\pi(s_\pm) t_{1,\pi}^1(s_\pm)}$$

Similar to before, we can write

$$F_{\pi,\pi}^V(s) \sim \frac{g_{\pi\pi\pi}}{g_{\pi\pi}} \frac{1}{s - s_0}$$

in the vicinity of the pole; in order to fix the

Additional kinematic factors, we demand the form factor to be reproduced by the Lagrangians in the narrow-width and $SU(2)$ limit, $g_{S\gamma} = \frac{g_{SM}}{3}$:



$$L_{W\gamma} = -\frac{e k_W^2}{g_{W\gamma}} A_{W\gamma}^{\mu\nu}$$

$$\mapsto V(W \rightarrow \gamma) = -\frac{e k_W^2}{g_{W\gamma}} E_{\gamma}^{\mu} E_{W\nu}$$

$$\mapsto V(S \rightarrow \gamma) = -\frac{e k_S^2}{g_{S\gamma}} E_{\gamma}^{\mu} E_{S\nu}$$

$$\begin{aligned} \mapsto V(W \rightarrow \gamma) &= -i \frac{e g_{SM} M_S^2}{g_{S\gamma}} (p_2 - p_1)_{\mu} \frac{i(-g^{\mu\nu} + \frac{p_S^{\mu} p_S^{\nu}}{M_S^2})}{p_S^2 - M_S^2 + i\epsilon} E_{\gamma}^{\nu} \\ &= -\frac{e g_{SM} M_S^2}{g_{S\gamma}} (p_2 - p_1)_{\mu} E_{\gamma}^{\mu} \frac{1}{p_S^2 - M_S^2 + i\epsilon} \end{aligned}$$

Again (as for the $\pi\pi$ scattering case), a little arbitrariness has to be fixed by the signs and factors of i , in particular because $F_{\pi}(s)$ can be defined with an additional factor of i !

where the pion-vector form factor is defined via

$$\langle \pi^+(p_2) | J_{\mu}(0) | 0 \rangle = e (p_2 - p_1)_{\mu} F_{\pi}^V(s)$$

(including the additional factor of e),

$$\text{so that } F_{\pi}^V(s) = \frac{g_{SM}}{g_{S\gamma}} \frac{M_S^2}{s - s_0}$$

in the limit $\Gamma_S \rightarrow 0$ and thus

$$F_{\pi,II}^V(s) = \frac{g_{SM}}{g_{S\gamma}} \frac{s_0}{s - s_0}$$

Altogether, we then have

since $t_{\pi,II}^1$ for $t_{\pi,II}$

$$\begin{aligned} F_{\pi,II}^V(s_-) &= \frac{F_{\pi,II}^V(s_-)}{1 - 2\alpha^{\pi}(s_-) t_{\pi,II}^1(s_-)} \stackrel{\downarrow}{=} \frac{F_{\pi,II}^V(s_-)}{1 - 2\alpha^{\pi}(s_-) \frac{t_{\pi,II}(s_-)}{1 + 2\alpha^{\pi}(s_-) t_{\pi,II}^1(s_-)}} \\ &= \frac{1 + 2\alpha^{\pi}(s_-) t_{\pi,II}^1(s_-) - 2\alpha^{\pi}(s_-) t_{\pi,II}^1(s_-)}{1 + 2\alpha^{\pi}(s_-) t_{\pi,II}^1(s_-)} \\ &= F_{\pi,II}^V(s_-) \left\{ 1 + 2\alpha^{\pi}(s_-) t_{\pi,II}^1(s_-) \right\} \end{aligned}$$

Radiative Resonance Couplings $\pi \rightarrow \pi^* \delta$

$$\mapsto F_{\pi, \pi}(s) = \frac{F_{\pi, \pi}^v(s)}{1 + 2\sigma^{\pi}(s) t_{\pi, \pi}^{\pi}(s)}$$

$$\mapsto F_{\pi, \pi}(s_3) = \frac{F_{\pi, \pi}^v(s_3)}{1 + 2\sigma^{\pi}(s_3) t_{\pi, \pi}^{\pi}(s_3)}$$

$$= \frac{\frac{g_{\pi\pi\pi}}{g_{33}} \frac{s_3}{s_3 - s}}{1 + 2\sigma^{\pi}(s_3) \frac{g_{\pi\pi\pi}^2 (s - 4M_{\pi}^2)}{48\pi (s_3 - s)}} \Big|_{s=s_3}$$

$$= \frac{\frac{g_{\pi\pi\pi}}{g_{33}} s_3}{(s_3 - s) + 2\sigma^{\pi}(s_3) \frac{g_{\pi\pi\pi}^2 (s - 4M_{\pi}^2)}{48\pi}} \Big|_{s=s_3}$$

$$= \frac{1}{\frac{g_{\pi\pi\pi}}{g_{33}}} \frac{48\pi s_3}{2\sigma^{\pi}(s_3) (s - 4M_{\pi}^2)} \Big|_{s=s_3}$$

$$\mapsto \frac{1}{g_{\pi\pi\pi} g_{33}} = \frac{2\sigma^{\pi}(s_3) (s_3 - 4M_{\pi}^2)}{48\pi s_3} F_{\pi, \pi}^v(s_3)$$

$$= \frac{i\sigma^{\pi}(s_3)}{24\pi} \left(1 - \frac{4M_{\pi}^2}{s_3}\right) = i \frac{\sigma^{\pi}(s_3)^3}{24\pi} F_{\pi, \pi}^v(s_3)$$

Note that although $\text{Im}s_3 < 0$, so that the representation with s should indeed be the proper one, we can use the other representation to obtain the same result (which is certainly less formal, as it involves an evaluation at s_i with $\text{Im}s_3 < 0$).

For the unitarity relation of $\pi \rightarrow \pi^*$,

$$\text{Im}f_{\pi}(s) = \sigma_{\pi}(s) (t_{\pi}^{\pi}(s))^* f_{\pi}(s) \theta(s - 4M_{\pi}^2)$$

$$\Leftrightarrow f_{\pi}(s_i) - f_{\pi}(s-1) = 2i\sigma_{\pi}(s) (t_{\pi}^{\pi}(s))^* f_{\pi}(s),$$

We can proceed similarly to find the analytic continuation, finding $(l=1)$

$$f_{1,II}(s_-) - f_{1,II}(s_+) = \mp 2\sigma^\pi(s_\pm) \underbrace{(t_{1,II}^1(s_\pm))^*}_{= t_{1,II}^1(s_-)} \underbrace{f_1(s_+)}_{= f_{1,II}(s_-)}$$

$$\Rightarrow f_{1,II}(s_-) = \frac{f_{1,II}(s_-)}{1 + 2\sigma^\pi(s_\pm) t_{1,II}^1(s_-)}$$

$$f_{1,II}(s_+) - f_{1,II}(s_-) = \mp 2\sigma^\pi(s_\pm) \underbrace{t_{1,II}^1(s_+)}_{= (t_{1,II}^1(s_+))^*} f_{1,II}(s_+)$$

$$\Rightarrow f_{1,II}(s_+) = \frac{f_{1,II}(s_+)}{1 + 2\sigma^\pi(s_\pm) t_{1,II}^1(s_+)}$$

In order to match onto

$$f_{1,II}^{VMD}(s) = \frac{2eg_{\sigma\pi\pi}g_{\sigma\pi\pi}}{M_\sigma^2 - iM_\sigma\Gamma_\sigma - s}$$

in the VMD limit, we can write

$$f_{1,II}(s) = \frac{2eg_{\sigma\pi\pi}g_{\sigma\pi\pi}}{s_0 - s}$$

in the vicinity of the pole. We then find

$$f_{1,II}(s_-) = \frac{f_{1,II}(s_-)}{1 - 2\sigma^\pi(s_-) t_{1,II}^1(s_-)} = \frac{f_{1,II}(s_-)}{1 - 2\sigma^\pi(s_-) \frac{t_{1,II}^1(s_-)}{1 + 2\sigma^\pi(s_-) t_{1,II}^1(s_-)}} = \frac{1 + 2\sigma^\pi(s_-) t_{1,II}^1(s_-) - 2\sigma^\pi(s_-) t_{1,II}^1(s_-)}{1 + 2\sigma^\pi(s_-) t_{1,II}^1(s_-)} = f_{1,II}(s_-) \left\{ 1 + 2\sigma^\pi(s_-) t_{1,II}^1(s_-) \right\}$$

$$\Rightarrow f_{1,II}(s_-) = \frac{f_{1,II}(s_-)}{1 + 2\sigma^\pi(s_-) t_{1,II}^1(s_-)}$$

$$\Rightarrow f_{1,II}(s_0) = \frac{2eg_{\sigma\pi\pi}g_{\sigma\pi\pi}}{s_0 - s} \bigg|_{s=s_0} = \frac{2eg_{\sigma\pi\pi}g_{\sigma\pi\pi}}{(s_0 - s) + 2\sigma^\pi(s_0) \frac{g_{\sigma\pi\pi}^2 (s - 4M_\pi^2)}{4g_\pi}} \bigg|_{s=s_0}$$

$$= \frac{4g_\pi eg_{\sigma\pi\pi}g_{\sigma\pi\pi}}{\sigma^\pi(s_0) g_{\sigma\pi\pi}^2 (s - 4M_\pi^2)} \bigg|_{s=s_0}$$

$$\Rightarrow \frac{eg_{\sigma\pi\pi}}{g_\pi} = \frac{\sigma^\pi(s_0) (s - 4M_\pi^2)}{4g_\pi} f_{1,II}(s) \bigg|_{s=s_0} = \frac{i\sigma^\pi(s_0) s \sigma^\pi(s)}{4g_\pi} f_{1,II}(s) = i \frac{s_0 \sigma^\pi(s_0)}{4g_\pi} f_{1,II}(s_0)$$

Note that the analytic continuation is "less straight-forward" than due to the pole had cut in $s \rightarrow s_+$; instead, one could use the discontinuity equation in the form

$f_{1,II}(s) - f_{1,II}(s) = -2\sigma^\pi(s) t_{1,II}^1(s)$ to obtain the "final" result. (Can also be done like this for the other relations)

Add. note to $f(s, t, u) = f(s) + \dots$ this is usually the starting point for Chiral-Triplets approximations, where we truncate the PA at some l , while

Add. note: eq. (13) can indeed be rewritten using the formula from the PDE for resonance formation (l. 50.)

are not exact, only chopping terms in the Bales program; let eqs. be used to

be used to