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Asymptotic Behavior of Meson Transition Form Factors 1

21.09.2020

In the following, we will check some of the calculations from Marhi's and Peter's paper. (see above the table make fit)

(see arXiv:2004.06127)

The asymptotic behavior of pseudoscalar transition form factors (describing the decay $P \rightarrow \gamma^*(q_1) \gamma^*(q_2)$) has been studied in the literature using an expansion along the light cone $x^2=0$.

At leading order, the resulting TFF for the pion can be expressed as

?
? where from?

$$F_{\pi \gamma \gamma}(q_1^2, q_2^2) = -\frac{2F_\pi}{3} \int_0^1 du \frac{\phi_\pi(u)}{uq_1^2 + (1-u)q_2^2} + \mathcal{O}(q_i^{-4})$$

in terms of the decay constant $F_\pi = 92,28$ (19) MeV and the wave function $\phi_\pi(u)$. The asymptotic form of the wave function reads $\phi_\pi(u) = 6u(1-u)$, resulting in

$$F_{\pi \gamma \gamma}(q^2, q^2) = -\frac{2F_\pi}{3q^2} + \mathcal{O}(q^{-4})$$

for the symmetric limit ($\hat{=}$ kinematic configuration that follows from a strict Operator Product expansion) and

$$F_{\pi \gamma \gamma}(q^2, 0) = -\frac{2F_\pi}{q^2} + \mathcal{O}(q^{-4})$$

for the singly-virtual case (often referred to as the Brodsky-Lepage limit of the singly-virtual TFF).

Here, we used that

$$\int_0^1 du [6u(1-u)] = 1, \quad \int_0^1 du [6(1-u)] = 3.$$

Note that the wave-function approach already retains higher-order terms, thus going beyond a strict OPE.

The Lorentz structure and helicity amplitudes have been worked out (not entirely) in the Mathematica - File.

We now come to the BL limit for the transition form factors in more detail.

Starting with the pseudoscalar case, where we restrict the analysis to the leading-order result, we define the decay constant F_P^a via

$$\langle 0 | \bar{q}(0) \gamma_5 \frac{\lambda_a}{2} q(0) | P(p) \rangle = i p_\mu F_P^a$$

P-odd

P.S. 07.10.2020: Need proper parity on LHS and RHS, i.e. γ_5 for pseudoscalar and $\gamma_5 \gamma_\mu$ for axial-vector later, also have matrix element with vector current.

Why don't we also define another (second) decay constant with the pseudoscalar current (as for scalar particles) → Discussion will Mahan probably exists for general PS that are not massless

with flavor decomposition using the Gell-Mann matrices λ_a and $\lambda_0 = \sqrt{3} T_8$. The wave functions $\phi_P^a(u)$ are defined as

$$\langle 0 | \bar{q}(x) \gamma_5 \frac{\lambda_a}{2} q(0) | P(p) \rangle = i p_\mu F_P^a \int_0^1 du e^{-i u p \cdot x} \phi_P^a(u)$$

where, however, an epsilon tensor appears on RHS. (not enough quantities to contract with ϵ -tensor)

where a path-ordered gauge factor to connect the quark fields at 0 and x has been omitted on the LHS.

Using conformal symmetry of QCD, the wave functions can be calculated asymptotically, resulting in

$$\phi_P^a(u) = 6u(1-u) \equiv \phi(u)$$

01.10.2020: Note that in the SU(3) limit (neglecting breaking effects), the LCDAs of all particles (charm) are trivially related (equivalent?); for SU(2) case e.g. charged and neutral

What path-ordered gauge factor to connect the quark fields at 0 and x? $[x, y] = P \exp \int_0^1 dt (x-y) \cdot A(t)$ $\frac{d}{dt} \ln \langle x, y | P \rangle = - \int_0^1 dt A(t) \cdot (x-y)$ - What for why? What's + needed to retain local gauge sym. mbr. V is gluon background field.

In fact, we will only consider asymptotic results here; to the extent possible, we will write the corresponding wave function in terms of $\phi(u)$ from above. Beyond the asymptotic result, the matrix element (*) and thus wave function become scale dependent. However, the conformal analysis shows that the higher-order terms can be organized in an expansion in Gegenbauer polynomials $C_n^{3/2}$:

$$\phi(u, \mu) = 6u(1-u) \sum_{n=0}^{\infty} a_n(\mu) C_n^{3/2}(2u-1),$$

$$a_0 = 1, \quad a_n(\mu) = a_n(\mu_0) \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_n/\beta_0}$$

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23.09.2020

where
$$\alpha_n^{(1)} = C_F \left(1 - \frac{2}{(n+1)(n+2)} + 4 \sum_{m=2}^{n+1} \frac{1}{m} \right),$$

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f,$$

$$C_F = \frac{N_c^2 - 1}{2N_c}.$$

Since $C_0^{3/2} = 1$ and due to the orthogonality relation

$$\int_0^1 du u(1-u) C_n^{3/2}(2u-1) C_m^{3/2}(2u-1) = \delta_{nm} \frac{(n+1)(n+2)}{4(2n+3)},$$

thus in particular

$$\int_0^1 du u(1-u) C_n^{3/2}(2u-1) \cdot 1 = \delta_{n0} \frac{1}{6},$$

$$= C_0^{3/2}(2u-1)$$

we find that the expansion is normalized according to

$$\int_0^1 du \phi(u, \mu) = \int_0^1 du \left\{ C_0(1-u) \sum_{n=0}^{\infty} a_n(\mu) C_n^{3/2}(2u-1) \right\}$$

$$= C_0 \sum_{n=0}^{\infty} a_n(\mu) \int_0^1 du u(1-u) C_n^{3/2}(2u-1) = a_0(\mu) = 1.$$

One can show that $\phi_p^a(u) = \eta(a) \phi_{pc}^a(1-u)$, where

$$\eta(a) = \begin{cases} +1 & \text{for } a = 0, 1, 3, 4, 6, 8 \\ -1 & \text{for } a = 2, 5, 7 \end{cases},$$

i.e. $\eta(a) = -1$ for complex Gell-Mann matrices λ^a and $\eta(a) = +1$ for real matrices. In particular, this follows from charge-conjugation and translation invariance,

$$\langle 0 | \bar{q}(x) \gamma_5 \frac{\lambda^a}{2} q(0) | P(p) \rangle = i p_\mu F_P^a \int_0^1 du e^{-i u p \cdot x} \phi_p^a(u)$$

$$\stackrel{!}{=} i p_\mu F_{Pc}^a \int_0^1 du e^{-i u p \cdot x} \phi_{pc}^a(1-u) \eta(a)$$

$$\stackrel{F_{Pc}^a = F_P^a}{=} i p_\mu F_P^a \int_0^1 du e^{-i(1-u)p \cdot x} \phi_{pc}^a(u) \eta(a)$$

$$= i p_\mu F_P^a e^{-i p \cdot x} \int_0^1 du e^{i u p \cdot x} \phi_{pc}^a(u) \eta(a)$$

How to prove with C-conj. and transl. inv. use C operator in vacuum and $\langle P(p) |$, see \square of



$$\begin{aligned}
 &= \eta(a) e^{-ipx} \left\{ i p_\mu F_P^a \int_0^1 du e^{iupx} \phi_{PC}^a(u) \right\} \\
 &= \eta(a) e^{-ipx} \langle 0 | \bar{q}(-x) \gamma_\mu \gamma_5 \int_0^1 q(u) | PC(p) \rangle \\
 &= e^{-ipx} \langle 0 | e^{-ix\hat{P}} \bar{q}(0) e^{ix\hat{P}} \gamma_\mu \gamma_5 \frac{(a)^x}{2} e^{-ix\hat{P}} q(x) e^{ix\hat{P}} | PC(p) \rangle \\
 &= \langle 0 | \bar{q}(0) \gamma_\mu \gamma_5 \frac{(a)^x}{2} q(x) | PC(p) \rangle = \langle 0 | C \bar{q}(x) \gamma_\mu \gamma_5 \frac{1}{2} q(0) C | PC(p) \rangle \\
 &= \langle 0 | \bar{q}(0) \gamma_\mu \gamma_5 \frac{1}{2} q(0) | PC(p) \rangle = \langle 0 | \bar{q}(0) \gamma_\mu \gamma_5 \frac{1}{2} q(0) | PC(p) \rangle
 \end{aligned}$$

to see this, do e.g. component calculation (in flavor space) of $C \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \gamma_\mu \gamma_5 \begin{pmatrix} u \\ d \end{pmatrix} C$

↓
 $\frac{1}{2} \frac{q(x) \lambda^a q(0)}{q(0) \lambda^a q(x)}$ in flavor space

A direct consequence for $P=PC$ and $a \in \{0, 1, 3, 4, 6, 8\}$ is that $\phi_P^a(u) = \phi_P^a(1-u)$, so that the odd coefficients in the Gegenbauer expansion vanish:

$$\begin{aligned}
 \phi(u, p) &= b u(1-u) \sum_{n=0}^{\infty} a_n(p) C_n^{3/2}(2u-1) \\
 \phi(1-u, p) &= b u(1-u) \sum_{n=0}^{\infty} a_n(p) C_n^{3/2}(1-2u)
 \end{aligned}$$

the $a_n(p)$ (in even for n even; odd for n odd) ?

For $a=2, 5, \dots$ only odd terms

i.e. $C_n^{3/2}(2u-1) \stackrel{!}{=} C_n^{3/2}(1-2u)$ or in other words only even terms.

The leading diagrams in the BLM formalism are obtained from contracting the quark fields in the time-ordered product using free propagators, resulting in $(q = \begin{pmatrix} u \\ d \end{pmatrix}, Q = \frac{1}{3} \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix})$

$$\begin{aligned}
 T \{ j_{em}^\mu(x) j_{em}^\nu(0) \} &= T \{ \bar{q}(x) Q \gamma^\mu q(x) \bar{q}(0) Q \gamma^\nu q(0) \} \\
 &\stackrel{\text{Wick's theorem}}{=} \bar{q}(x) Q^2 \gamma^\mu \overline{q(x) \bar{q}(0)} \gamma^\nu q(0) + \bar{q}(0) Q^2 \gamma^\nu \overline{q(0) \bar{q}(x)} \gamma^\mu q(x) \\
 &= \bar{q}(x) Q^2 \gamma^\mu \gamma^\alpha \gamma^\nu q(0) S_\alpha^F(x) + \bar{q}(0) Q^2 \gamma^\nu \gamma^\alpha \gamma^\mu q(x) S_\alpha^F(-x) \quad (*)
 \end{aligned}$$

where $S_p^F(x) = i \int \frac{d^4p}{(2\pi)^4} \frac{p_\mu e^{-ipx}}{p^2 + i\epsilon} = \frac{i x_\mu}{2\pi^2 (x^2 - i\epsilon)^2}$, in general, i.e. into, a very complicated expression.

the fermionic propagator in position space (see internet). Note that in using Wick's theorem, we kept exactly those terms which have \bar{q}, q left in normal-ordered form, since these are the relevant ones for $\langle 0 | PC(p) \rangle$ etc. - otherwise - vanish

$$M_{PC}^{\mu\nu}(p \rightarrow q_1, q_2) = i \int d^4x e^{iq_1 \cdot x} \langle 0 | T \{ j_{em}^\mu(x) j_{em}^\nu(0) \} | PC(p) \rangle$$

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27.09.2025 For the remaining Dirac Structure, we use

$$\gamma^\mu \gamma^\alpha \gamma^\nu = g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha + i \epsilon^{\mu\alpha\nu\beta} \gamma_5,$$

so that

$$\begin{aligned} T \{ j_{em}^\mu(x) j_{em}^\nu(0) \} &= \bar{q}(x) Q^2 \left[g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha + i \epsilon^{\mu\alpha\nu\beta} \gamma_5 \right] \\ &\quad \times q(0) S_\alpha^F(x) \\ &\quad + \bar{q}(0) Q^2 \left[g^{\nu\alpha} \gamma^\mu + g^{\mu\alpha} \gamma^\nu - g^{\mu\nu} \gamma^\alpha + i \epsilon^{\nu\alpha\mu\beta} \gamma_5 \right] \\ &\quad \times q(x) S_\alpha^F(-x). \end{aligned}$$

$$\begin{aligned} \Rightarrow T^{\mu\nu}(p \rightarrow q_1, q_2) &= i \int d^4x e^{iq_1 x} \langle 0 | \bar{q}(x) Q^2 \left[g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha + i \epsilon^{\mu\alpha\nu\beta} \gamma_5 \right] \\ &\quad \times q(0) S_\alpha^F(x) \\ &\quad + \bar{q}(0) Q^2 \left[g^{\nu\alpha} \gamma^\mu + g^{\mu\alpha} \gamma^\nu - g^{\mu\nu} \gamma^\alpha - i \epsilon^{\nu\alpha\mu\beta} \gamma_5 \right] \\ &\quad \times q(x) S_\alpha^F(-x) | P(p) \rangle \end{aligned}$$

$$\begin{aligned} &= i \int d^4x e^{iq_1 x} \langle 0 | \bar{q}(x) Q^2 \left[g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha + i \epsilon^{\mu\alpha\nu\beta} \gamma_5 \right] \\ &\quad \times q(0) S_\alpha^F(x) \\ &\quad + e^{-i\hat{p}x} \bar{q}(0) e^{i\hat{p}x} Q^2 \left[g^{\nu\alpha} \gamma^\mu + g^{\mu\alpha} \gamma^\nu - g^{\mu\nu} \gamma^\alpha - i \epsilon^{\nu\alpha\mu\beta} \gamma_5 \right] \\ &\quad \times e^{-i\hat{p}x} q(x) e^{i\hat{p}x} e^{-i\hat{p}x} S_\alpha^F(-x) | P(p) \rangle \end{aligned}$$

$$\begin{aligned} &= i \int d^4x e^{iq_1 x} \langle 0 | \bar{q}(x) Q^2 \left[g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha + i \epsilon^{\mu\alpha\nu\beta} \gamma_5 \right] \\ &\quad \times q(0) S_\alpha^F(x) \\ &\quad + \bar{q}(0) Q^2 \left[g^{\nu\alpha} \gamma^\mu + g^{\mu\alpha} \gamma^\nu - g^{\mu\nu} \gamma^\alpha - i \epsilon^{\nu\alpha\mu\beta} \gamma_5 \right] \\ &\quad \times q(x) e^{-i\hat{p}x} S_\alpha^F(-x) | P(p) \rangle \end{aligned}$$

$$\begin{aligned} &= i \int d^4x e^{iq_1 x} \langle 0 | \bar{q}(x) Q^2 (i \epsilon^{\mu\alpha\nu\beta} \gamma_5) q(0) S_\alpha^F(x) \\ &\quad + e^{-i\hat{p}x} \bar{q}(-x) Q^2 (-i \epsilon^{\nu\alpha\mu\beta} \gamma_5) q(0) [-S_\alpha^F(x)] | P(p) \rangle \\ &\quad \langle 0 | \bar{q}(x) \gamma_\mu \gamma_5 \frac{\not{x}_0}{2} q(0) | P(p) \rangle = \frac{1}{2} \langle P | F_p^a \int du e^{-i u \cdot x} \phi_p^a(u) \end{aligned}$$

In paper: using commutational line and the symmetry of the wave function under $u \rightarrow 1-u$, all contractions (see previous page) field the same result... before need what matrix element vector current vertices?

Matrix element of vector current

vertices

So that

$$e^{-ipx} \langle 0 | \bar{q}(x) \gamma_5 q(x) | P(p) \rangle = i p_\mu F_P^a \int_0^1 du e^{-i(1-u)p \cdot x} \phi_P^a(u)$$

$$\stackrel{\phi_P^a(u) = \phi_P^a(1-u)}{\downarrow} = i p_\mu F_P^a \int_0^1 du e^{-iup \cdot x} \phi_P^a(u) = \langle 0 | \bar{q}(x) \gamma_5 \frac{\lambda_a}{2} q(0) | P(p) \rangle$$

$$= i \int d^4x e^{iq \cdot x} (2i \epsilon_{\alpha\beta}^{\mu\nu}) \langle 0 | \bar{q}(x) Q^2 \gamma^\beta \gamma_5 q(0) | P(p) \rangle S_F^\alpha(x)$$

from translational invariance and the symmetry of the wave function under $u \rightarrow 1-u$

Using $Q^2 = a_0 \lambda^0 + a_i \lambda^i \rightarrow a_0 = \text{Tr}[\lambda^0 Q^2] \cdot \frac{3}{2} \cdot \frac{1}{3}$,
 $a_i = \text{Tr}[\lambda^i Q^2] \cdot \frac{1}{2}$,

we find

$$\langle 0 | \bar{q}(x) Q^2 \gamma^\beta \gamma_5 q(0) | P(p) \rangle = \sum_a C_a \langle 0 | \bar{q}(x) \gamma^\beta \gamma_5 \lambda_a q(0) | P(p) \rangle$$

and thus

$$C_a = \frac{1}{2} \text{Tr}[\lambda^a Q^2] = \frac{1}{2} \text{Tr}[Q^2 \lambda^a]$$

$$M_{\mu\nu} = i \int d^4x e^{iq \cdot x} (2i \epsilon_{\mu\nu\alpha\beta}) \langle 0 | \bar{q}(x) Q^2 \gamma^\beta \gamma_5 q(0) | P(p) \rangle S_F^\alpha(x)$$

$$= -4i \sum_a C_a F_P^a \underbrace{(q_1 + q_2)^\beta}_{=p^\beta} \int_0^1 du \phi_P^a(u) \int d^4x e^{iq_1 \cdot x} e^{-iup \cdot x} S_F^\alpha(x)$$

At this, $C_3 = \frac{1}{6}$, $C_8 = \frac{1}{6\sqrt{3}}$, $C_0 = \frac{2}{3\sqrt{6}}$, and all other vanish.

Since $S_F^\mu(x) = i \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu e^{-ipx}}{p^2 + i\epsilon}$, we find for the Feynman Propagator that

$$\int d^4x S_F^\mu(x) e^{iq \cdot x} = i \frac{q^\mu}{q^2}$$

just by Fourier transforming.

Furthermore,

$$\int d^4x x^\mu S_F^\nu(x) e^{iq \cdot x} = -i \partial_q^\mu \int d^4x S_F^\nu(x) e^{iq \cdot x}$$

$$= -i \partial_q^\mu \left[i \frac{q^\nu}{q^2} \right] = \frac{g^{\mu\nu}}{q^2} - \frac{2q^\mu q^\nu}{q^4}$$

as well as

$$\int d^4x x^\mu x^\nu S_F^\alpha(x) e^{iq \cdot x} = -i \partial_q^\mu \int d^4x x^\nu S_F^\alpha(x) e^{iq \cdot x}$$

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$$= -i \partial_j^v \left[\frac{g^{\mu\lambda}}{q^2} - \frac{2q^\mu q^\lambda}{q^4} \right] = \frac{+2iq^\mu g^{\nu\lambda}}{q^4} + \frac{2ig^{\mu\nu} q^\lambda}{q^4} + \frac{2ig^{\lambda\mu} q^\nu}{q^4} - \frac{8iq^\mu q^\lambda q^\nu q^2}{q^8}$$

$$= \frac{2i}{q^4} \left(g^{\mu\nu} q^\lambda + g^{\mu\lambda} q^\nu + g^{\lambda\mu} q^\nu - \frac{4q^\mu q^\lambda q^\nu}{q^2} \right).$$

The matrix element is then calculated to be ($q = q_1 = -q_2$)

$$M_{\mu\nu} = 4 \sum_a C_a F_P^a E_{\mu\nu\alpha\beta} (q_1 + q_2)^\beta \int_0^1 du \phi(u) \frac{q_1^\alpha}{q^2}$$

$$\stackrel{P=q_1+q_2}{=} 4 \sum_a C_a F_P^a E_{\mu\nu\alpha\beta} q_2^\beta \int_0^1 du \phi(u) \frac{q_1^\alpha (1-u)}{q^2}$$

$$- 4 \sum_a C_a F_P^a E_{\mu\nu\alpha\beta} q_1^\beta \int_0^1 du \phi(u) \frac{u q_2^\alpha}{q^2}$$

$$E_{\text{antisym}} \stackrel{=} 4 \sum_a C_a F_P^a E_{\mu\nu\alpha\beta} q_1^\nu q_2^\beta \int_0^1 du \frac{\phi(u)}{q_1^2 + u^2 m_P^2 - \underbrace{2u(q_1^2 + q_1 \cdot q_2)}_{=-2uq_1^2 - u(m_P^2 - q_1^2 - q_2^2)}}$$

$$E_{\text{sym}} \stackrel{=} -4 \sum_a C_a F_P^a E_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \int_0^1 du \frac{\phi(u)}{(1-u)q_1^2 + uq_2^2 - u(1-u)m_P^2}$$

Comparing this with

$$M_{\mu\nu} = E_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{P\text{sym}}(q_1^2, q_2^2)$$

leads to the transition form factor

$$F_{P\text{sym}}(q_1^2, q_2^2) = -4 \sum_a C_a F_P^a \int_0^1 du \frac{\phi(u)}{uq_1^2 + (1-u)q_2^2 - u(1-u)m_P^2}$$

\uparrow
 $u \rightarrow 1-u$ and $\phi(u) = \phi(1-u)$,

which reproduces the asymptotic behavior from the beginning of these notes

Note that although the mass m_P was kept in the final result, this leading-order derivation does not provide a consistent

Treatment of mass effects. To this end, one would have to differentiate between the meson momentum p and the light-cone momentum $k_\mu = p_\mu - x_\mu \frac{m^2}{2p \cdot x}$, which would appear in the integral of the wave packet $\phi^a(x)$ (in the e^{-iipx} factor); accordingly, including terms of $\mathcal{O}(m^2)$ would require the consideration of subleading terms in the light-cone expansion.

Moreover, the obtained result can only be strictly justified from an OPE in the limit in which both photon virtualities are large.

We now turn to scalar mesons, where we equivalently define a decay constant for the vector and scalar current:

$$\begin{aligned} \langle 0 | \bar{q}(0) \gamma_\mu \frac{\lambda^a}{2} q(0) | S(p) \rangle &= -p_\mu F_S^a \\ \langle 0 | \bar{q}(0) \frac{\lambda^a}{2} q(0) | S(p) \rangle &= m_S \bar{F}_S^a(p), \end{aligned}$$

where the scale dependence in $\bar{F}_S^a(p)$ is canceled by the one of the quark masses. In particular, the two decay constants are related by conservation of the vector current, i.e.

$$\begin{aligned} \partial_\mu V^{\mu a} &= i \bar{q} \left[M, \frac{\lambda^a}{2} \right] q, \quad V^{\mu a} = \bar{q} \gamma^\mu \frac{\lambda^a}{2} q, \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \\ \left(\partial_\mu A^{\mu a} = i \bar{q} \left[\frac{\lambda^a}{2}, M \right] \gamma_5 q, \quad A^{\mu a} = \bar{q} \gamma^\mu \gamma_5 \frac{\lambda^a}{2} q \right. \end{aligned}$$

Similarly relates the axial-vector and pseudoscalar - isovector-current; for non-vanishing quark and/or pion masses, this becomes relevant for the pion decay constant as well.

So that

$$\partial^\mu \langle 0 | \bar{q}(0) \gamma_\mu \frac{\lambda^a}{2} q(0) | S(p) \rangle = i p^2 F_S^a = i m_S^2 F_S^a$$

Using the divergence from above, we can alternatively

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29.09.2020 Calculate

$$\text{Im}_s^2 F_s^a = i \langle 0 | \bar{q}(0) [M, \frac{\lambda^a}{2}] q(0) | S(p) \rangle$$

$$\left\{ \begin{aligned} M &= a_0 \lambda^0 + a_i \lambda^i, & a_0 &= \text{Tr}[\lambda^0 M] \cdot \frac{1}{2} \\ & & a_i &= \text{Tr}[\lambda^i M] \cdot \frac{1}{2} \end{aligned} \right.$$

$$= i \sum_b a_b \langle 0 | \bar{q}(0) [\lambda^b, \frac{\lambda^a}{2}] q(0) | S(p) \rangle$$

$$\left\{ [\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c \right.$$

$$= i \sum_b a_b \langle 0 | \bar{q}(0) (i f^{bac} \lambda^c) q(0) | S(p) \rangle$$

$$= i \sum_b i f^{bac} \text{Tr}[\lambda^b M] \langle 0 | \bar{q}(0) \frac{\lambda^c}{2} q(0) | S(p) \rangle$$

$$= i \sum_c i f^{cab} \text{Tr}[\lambda^c M] m_s \bar{F}_s^b(\mu)$$

$$= i \sum_c i f^{abc} \bar{F}_s^b(\mu) m_s \text{Tr}[M \lambda^c]$$

$$\Rightarrow F_s^a = i f^{abc} \bar{F}_s^b(\mu) \frac{\text{Tr}[M \lambda^c]}{m_s}$$

Why odd terms only have for $a=0, 1, 3, 4, 6, 8$ and $P=PC?$

At this, $\sqrt{3}$

C	0	1	2	3	4	5	6	7	8
$\text{Tr}[M \lambda^c]$	$\sqrt{\frac{2}{3}}(m_u + m_d + m_s)$	0	0	$m_u - m_d$	0	0	0	0	$\frac{m_u + m_d - 2m_s}{\sqrt{3}}$

hence, $F_s^a = 0$ for $a=0, 3, 8$, in such a way that the leading term in the light-cone expansion vanishes.

Contrary to the pseudoscalar mesons, only odd powers in the Gegenbauer expansion contribute (see the two papers where the scalar meson wave functions are defined). The

Why do only odd powers in the Gegenbauer expansion contribute? as for pseudoscalars $\int du \phi_s^a(u, \mu) = 0 (= F_s^a)$

Normalization $\int du \phi_s^a(u, \mu) = 0 (= F_s^a)$

reflects the fact that $F_s^a = 0$. Therefore, the first non-vanishing term in the Gegenbauer expansion of $\phi_s^a(u, \mu)$ involves an unknown Gegenbauer coefficient; this coefficient can be made dimensionless by factoring out the scalar decay constant \bar{F}_s^a . We then write

$$\langle 0 | \bar{q}(x) \gamma_\mu \frac{\lambda^a}{2} q(0) | S(q) \rangle = -p_\mu \bar{F}_s^a(\mu) B_1(\mu) \int_0^1 du e^{-iup \cdot x} \underbrace{\frac{1}{3(2a-1)} \phi(u)}_{\text{the first vanishing term}}$$

$B_1(\mu)$ referring to the Gegenbauer coefficient and assuming that all flavor dependence is captured by $\bar{F}_s^a(\mu)$. Note that in contrast to the pseudoscalar case, the additional factor of $(2a-1)$ in the integral gives rise to an extra minus sign upon $u \rightarrow 1-u$ (see the derivation for the pseudoscalar case; this is also consistent with $\phi_s^a(u) = -\phi_s^a(1-u)$ from before, written with μ as a side note).

only $F_s^a = 0$ for $a=0,3,8$ regarding the normalization. OK at most for $a=0,1,3,4,6,8$ due to even terms $\frac{3(2a-1)}{2} \phi(u)$ (at $u=1$)
 Note that $\phi(u, 1-u)$ for $a=0,3,8$ are the only terms that matter due to Ca
 For different a the wave functions are different sym. prop $\phi^a(1-u) = \eta(a) \phi^a(u)$ so that F_s^a also needs to "catch" this in some way due to a -dependence but this is impossible? see above, only $a=0,3,8$ are relevant, which all have same symmetry

We are now ready to calculate the matrix element

$$M_{\mu\nu}^{\alpha\beta}(\phi \rightarrow q_1, q_2) = i \int d^4x e^{iq_1 \cdot x} \langle 0 | T \{ j_\mu^\alpha(x) j_\nu^\beta(0) \} | S(q) \rangle$$

$$= i \int d^4x e^{iq_1 \cdot x} \langle 0 | \bar{q}(x) Q^2 [g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha + i \epsilon^{\mu\nu\alpha\beta}] q(0) S_\alpha^F(x) + \bar{q}(0) Q^2 [g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha - i \epsilon^{\mu\nu\alpha\beta}] q(x) S_\alpha^F(-x) | S(q) \rangle$$

axial current vanishes and steps from PS =

$$i \int d^4x e^{iq_1 \cdot x} \langle 0 | \bar{q}(x) Q^2 [g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha] q(0) S_\alpha^F(x) - e^{-iix} \bar{q}(-x) Q^2 [g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha] q(x) S_\alpha^F(x) | S(q) \rangle$$

similar to PS but additional minus sign in wave function $\phi_s(u)$ upon $u \rightarrow 1-u$

$$= 2i \int d^4x e^{iq_1 \cdot x} \langle 0 | \bar{q}(x) Q^2 [g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha] q(0) S_\alpha^F(x) | S(q) \rangle$$

$$| Q^2 = C_a \lambda^a, C_a = \frac{1}{2} \text{Tr}[Q^2 \lambda^a]$$

$$= 4i \sum_a C_a \int d^4x e^{iq_1 \cdot x} \langle 0 | \bar{q}(x) [g^{\mu\alpha} \gamma^\nu + g^{\nu\alpha} \gamma^\mu - g^{\mu\nu} \gamma^\alpha] \frac{\lambda^a}{2} q(0) | S(q) \rangle S_\alpha^F(x)$$

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$$= 4i \sum_a C_a \int d^4x e^{iq \cdot x} \left\{ g^{\mu\alpha} p^\nu \bar{F}_S^a(\mu) B_1(\mu) \int_0^1 du e^{-iup \cdot x} 3(2u-1) \phi(u) \right. \\ \left. - g^{\nu\alpha} p^\mu \bar{F}_S^a(\mu) B_1(\mu) \int_0^1 du e^{-iup \cdot x} 3(2u-1) \phi(u) \right. \\ \left. + g^{\mu\nu} p^\alpha \bar{F}_S^a(\mu) B_1(\mu) \int_0^1 du e^{-iup \cdot x} 3(2u-1) \phi(u) \right\} S_\alpha^F(x)$$

$$= -4i \sum_a C_a \bar{F}_S^a(\mu) B_1(\mu) \int_0^1 du 3(2u-1) \phi(u) \left\{ g^{\mu\alpha} p^\nu + g^{\nu\alpha} p^\mu - g^{\mu\nu} p^\alpha \right\} \\ \times \int d^4x e^{ix(q_1 - up)} S_\alpha^F(x)$$

integrate part FT

$$\stackrel{\uparrow}{q=q_1-up} \frac{1}{4} 4 \sum_a C_a \bar{F}_S^a(\mu) B_1(\mu) \int_0^1 du \frac{3(2u-1) \phi(u)}{q^2} \left\{ g^{\mu\nu} p^\alpha + g^{\nu\mu} p^\alpha - g^{\mu\nu} (p \cdot q) \right\}$$

Note that this expression is - in contrast to the pseudoscalar case - only manifestly gauge invariant for $m_S=0$. To see this, we calculate

$$q_{1\mu} \mathcal{M}^{\mu\nu} \propto (q_1 \cdot q) (q_1^\nu + q_2^\nu) + (q_1 \cdot p) (q_1^\nu - u p_1^\nu) - (p \cdot q) q_1^\nu \\ = q_1^\nu \left[(q \cdot q_1) + (q_1 \cdot p) - u (q_1 \cdot p) - (p \cdot q) \right] + q_2^\nu \left[(q \cdot q_1) - u (p \cdot q_1) \right]$$

$$= q_1^\nu \left[(q \cdot q_2) + (p \cdot q_1) (1-u) \right] + q_2^\nu \left[q_1 \cdot (q - up) \right]$$

$$\left| \begin{aligned} (q \cdot q_2) &= (q_1 \cdot q_2) - u q_2^2 + (q_1 \cdot q_2) \\ &= (q_1 \cdot q_2) (1-u) - u q_2^2 \end{aligned} \right.$$

$$\left| \begin{aligned} (q_1 \cdot q_2) &= \frac{m_S^2 - q_1^2 - q_2^2}{2} \end{aligned} \right.$$

$$\left| \begin{aligned} (q_1 \cdot (q - up)) &= (q_1 \cdot (2q - q_1)) = 2(q \cdot q_1) - q_1^2 \\ &= q_1^2 - 2u (q_1^2 + (q_1 \cdot q_2)) = q_1^2 - u(m_S^2 + q_1^2 - q_2^2) \end{aligned} \right.$$

$$= q_1^\nu \left[(q_1^2 + (q_1 \cdot q_2)) (1-u) - (q_1 \cdot q_2) (1-u) + u q_2^2 \right]$$

$$q_1 + q_2 = p \quad + q_2^\nu \left[q_1^2 (1-u) + u q_2^2 - u m_S^2 \right]$$

$$\stackrel{\downarrow}{=} \left[q_1^2 (1-u) + u q_2^2 \right] p^\nu - u m_S^2 q_2^\nu$$

$$\begin{aligned}
 q^2 &= q_1^2 + u^2 p^2 - 2u(q_1 \cdot p) \\
 &= q_1^2 + u^2 p^2 - 2u \left(q_1^2 + \frac{m_s^2 - q_1^2 - q_2^2}{2} \right) \\
 &= q_1^2(1-u) + u^2 p^2 - u m_s^2 + u q_2^2
 \end{aligned}$$

$$\implies q_1^2(1-u) + u q_2^2 = q^2 - u^2 p^2 + u m_s^2$$

$$\begin{aligned}
 &= (q^2 - u^2 p^2 + u m_s^2) p^\nu - u m_s^2 q_2^\nu \\
 &= (q^2 - u^2 p^2) p^\nu + u m_s^2 q_1^\nu
 \end{aligned}$$

Inserting this back into the full expression for $\mathcal{M}^{\mu\nu}$, i.e. with the integral, reveals that we need $p^2 = m_s^2 = 0$ for the expression to be gauge invariant ($\hat{=}$ vanish); see also Mathematica. Analogously, we could show that $q_{2\nu} \mathcal{M}^{\mu\nu}$ only vanishes for $p^2 = m_s^2 = 0$.

The goal is now to project this onto the BTT structures

$$T_1^{\mu\nu} = (q_1 \cdot q_2) g^{\mu\nu} - q_1^\mu q_2^\nu$$

$$T_2^{\mu\nu} = q_1^2 q_2^2 g^{\mu\nu} + (q_1 \cdot q_2) q_1^\mu q_2^\nu - q_1^\mu q_2^\nu q_1^\nu - q_2^\mu q_1^\nu q_2^\nu$$

and read off the form factors in

$$\mathcal{M}^{\mu\nu} = \frac{1}{m_s} T_1^{\mu\nu} F_1^s + \frac{1}{m_s^3} T_2^{\mu\nu} F_2^s$$

Using Mathematica, we find that

$$\mathcal{M}_{\mu\nu} = 4 \sum_a C_a \bar{F}_s^a(\mu) B_1(\mu) \int_0^1 du \frac{3(2u-1)\phi(u)}{q^2} \left((2u-1)(T_1)_{\mu\nu} + \left[\frac{u}{q_1^2} + \frac{u-1}{q_2^2} \right] (T_2)_{\mu\nu} \right)$$

$$\begin{aligned}
 \implies F_1^s(q_1^2, q_2^2) &= 4 \sum_a C_a \bar{F}_s^a(\mu) B_1(\mu) m_s \int_0^1 du \frac{3(2u-1)\phi(u)}{q_1^2(1-u) + u q_2^2} \\
 q^2 &= q_1^2 + u^2 p^2 - 2u(q_1 \cdot p) = q_1^2 + u^2 m_s^2 - 2u \left(q_1^2 + \frac{m_s^2 - q_1^2 - q_2^2}{2} \right) \\
 &= q_1^2(1-u) + u q_2^2 + u^2 m_s^2 - u m_s^2 \quad (\text{and } m_s^2 = 0)
 \end{aligned}$$

Mass is assumed to vanish? or not in the prefactor? see also e-mail Marking 08.09.2020 14:24 o'clock.

$$F_2^s(q_1^2, q_2^2) = 4 \sum_a C_a \bar{F}_s^a(\mu) B_1(\mu) m_s^3 \int_0^1 du \frac{3(2u-1)\phi(u) [u q_2^2 + (1-u) q_1^2]}{[q_1^2(1-u) + u q_2^2] q_1^2 q_2^2}$$

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In paper it is stated that in the limit $m_s \rightarrow 0$, this direct projection onto the BTT structures produces a singularity at $q_1 \cdot q_2 = 0$ which I don't see where / how?

Use Mathematica and integration by parts with

$$u' = 3(2u-1)\phi(u), \quad \phi(u) = 6u(1-u)$$

$$V = \frac{u q_2^2 + (1-u) q_1^2}{[u q_2^2 + (1-u) q_1^2] q_1^2 q_2^2}$$

$$\int u'v = \underbrace{uV}_{=0} - \int u v'$$

$$= 4 \sum_c C_a \bar{F}_s^a(\mu) B_1(\mu) m_s^3 \int_0^1 du \frac{3u(1-u)\phi(u)}{[q_1^2(1-u) + u q_2^2]^2}$$

The axial-vector mesons are more difficult to handle in this formalism, where the main complication compared to the (pseudo-)scalar mesons is that the polarization vector contributes to different orders in the twist expansion; hence, at each order a different wave function may occur. Before going into more detail about this, we define the decay constants via

$$\langle \bar{q}(0) \gamma_\mu \gamma_5 \frac{\lambda^a}{2} q(0) | A(p, \Delta_A) \rangle = F_A^a m_A G_\mu.$$

Why not a part with P_μ but only the pol. vector (axial-vector?) for pseudo-scalar also had (only) P_μ or RHS ?

The aforementioned different orders are separated by defining a

light-cone vector

$$k_\mu = P_\mu - x_\mu \frac{m_A^2}{2(p \cdot x)},$$

which on the light-cone, $x^2 = 0$, fulfills

$$k^2 = p^2 + x^2 \frac{m_A^4}{4(p \cdot x)^2} - 2 \frac{m_A^2}{2} \frac{p^2 - m_A^2}{2} = 0.$$

Probably need the derivation of the expansion with ϵ^{12} to understand something ϵ^{12} ?

The polarization vector is then decomposed according to \perp on k and x

$$E^\mu = \frac{E \cdot k}{k \cdot x} k^\mu + \frac{E \cdot x}{k \cdot x} x^\mu + E_\perp^\mu \quad (\text{s.t. } E \cdot k = E \cdot k \text{ and } E \cdot x = E \cdot x)$$

$$E \cdot k = E \cdot p - E \cdot x \frac{m_A^2}{2(p \cdot x)} = -E \cdot x \frac{m_A^2}{2(p \cdot x)}$$

$$|k \cdot x = p \cdot x - x^2 \frac{m_A^2}{2(p \cdot x)} = p \cdot x \text{ for } x^2=0$$

$$= \frac{E \cdot x}{k \cdot x} \left(k^\mu - \frac{m_A^2}{2(k \cdot x)} x^\mu \right) + E_\perp^\mu$$

This decomposition gives rise to three different wave functions occurring in the axial-vector matrix element:

$$\langle 0 | \bar{q}(x) \gamma_5 \frac{\not{x}}{2} q(0) | A(p, x) \rangle = F_A^a m_A \int_0^1 du e^{-iuk \cdot x} \left[k^\mu \frac{E \cdot x}{k \cdot x} \phi(u) + E_\perp^\mu \phi_\perp(u) - x^\mu \frac{m_A^2 E \cdot x}{2(k \cdot x)^2} \phi_3(u) \right],$$

where $\phi_\perp(u)$ and $\phi_3(u)$ are of higher twist.

In order to obtain a gauge-invariant result for the TFF, these wave functions should be replaced by so-called Wandzura-Wilczek relations in terms of the leading twist-2 distribution amplitudes, which effectively neglects three-pion contributions.

In this approximation, one has

$$\phi_\perp(u) = \frac{1}{2} \left\{ \int_0^u dv \frac{\phi(v)}{1-v} + \int_u^1 dv \frac{\phi(v)}{v} \right\} \stackrel{\text{Mathematica}}{=} \frac{3}{2} + 3(u-1)u$$

$$= \frac{1}{2} (3 - \phi(u))$$

for the asymptotic $\phi(u)$ from the pseudoscalar mesons. The wave function $\phi_3(u)$ does not actually contribute due to the antisymmetry of the E -tensor - but it could be obtained with similar methods (see reference in paper).

How is $\phi_3(u)$ related to the E -tensor?
 see later step
 contraction of ϵ with $S^{\mu\nu}$ and x prefactor of $\phi_3(u)$ vanishes.

In contrast to the pseudoscalar case, there is now also a non-vanishing contribution from the vector matrix element

$$\langle 0 | \bar{q}(x) \gamma^\mu \frac{\not{x}}{2} q(0) | A(p, x) \rangle = -\frac{1}{4} F_A^a m_A E^{\mu\nu\alpha\beta} \epsilon_{\nu\alpha\beta\gamma} \int_0^1 du e^{-iuk \cdot x} \phi(u),$$

Why not also only γ_5 ?

which is again a twist-3 contribution and technically requires

Some F_A^a in A and V current?

Another wave function. In the same approximation as for $\phi_\perp(u)$

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from before, however, this new wave function becomes

$$2u(1-u) \int_0^u \frac{\phi(v)}{1-v} + 2u \int_u^1 \frac{\phi(v)}{v} = -6(u-1)u = \phi(u)$$

Mathematics

asymptotically.

Using the time-ordered product $T \{ j_{\mu}^{\dagger}(x) j_{\nu}(0) \}$ from before, we find

$$E_{\alpha} M^{\mu\nu} = i \int d^4x e^{i q \cdot x} \langle 0 | T \{ j_{\mu}^{\dagger}(x) j_{\nu}(0) \} | A(p, \lambda_A) \rangle$$

$$= i \int d^4x e^{i q \cdot x} \langle 0 | \bar{q}(x) \mathcal{Q}^2 [g^{\mu\alpha} \gamma^{\nu} + g^{\nu\alpha} \gamma^{\mu} - g^{\mu\nu} \gamma^{\alpha} + i \epsilon^{\mu\nu\alpha\beta} \gamma^{\beta} \gamma_5] q(0) S_{\alpha}^F(x) | A(p, \lambda_A) \rangle$$

$$+ \bar{q}(0) \mathcal{Q}^2 [g^{\mu\alpha} \gamma^{\nu} + g^{\nu\alpha} \gamma^{\mu} - g^{\mu\nu} \gamma^{\alpha} - i \epsilon^{\mu\nu\alpha\beta} \gamma^{\beta} \gamma_5] q(x) S_{\alpha}^F(-x) | A(p, \lambda_A) \rangle$$

$$= i \int d^4x e^{i q \cdot x} \langle 0 | \bar{q}(x) \mathcal{Q}^2 [g^{\mu\alpha} \gamma^{\nu} + g^{\nu\alpha} \gamma^{\mu} - g^{\mu\nu} \gamma^{\alpha} + i \epsilon^{\mu\nu\alpha\beta} \gamma^{\beta} \gamma_5] q(0) S_{\alpha}^F(x) - e^{-i p \cdot x} \bar{q}(-x) \mathcal{Q}^2 [g^{\mu\alpha} \gamma^{\nu} + g^{\nu\alpha} \gamma^{\mu} - g^{\mu\nu} \gamma^{\alpha} - i \epsilon^{\mu\nu\alpha\beta} \gamma^{\beta} \gamma_5] q(x) S_{\alpha}^F(-x) | A(p, \lambda_A) \rangle$$

$$\langle 0 | \bar{q}(0) \gamma^{\mu} \frac{\not{1}}{2} q(0) | A(p, \lambda_A) \rangle = -\frac{1}{4} F_A^a m_A E^{\mu\nu\alpha\beta} E_{\nu\kappa\alpha} x_{\beta} \int_0^1 du e^{-i u k \cdot x} \phi(u)$$

$$\langle 0 | \bar{q}(0) \gamma^{\mu} \gamma_5 \frac{\not{1}}{2} q(0) | A(p, \lambda_A) \rangle = F_A^a m_A \int_0^1 du e^{-i u k \cdot x} \left[k^{\mu} \frac{E \cdot x}{k \cdot x} \phi(u) + E_{\perp}^{\mu} \phi_{\perp}(u) \right]$$

neglecting $\phi_3(u)$

so that

$$e^{i p \cdot x} \langle 0 | \bar{q}(-x) \gamma^{\mu} \frac{\not{1}}{2} q(0) | A(p, \lambda_A) \rangle = \frac{1}{4} F_A^a m_A E^{\mu\nu\alpha\beta} E_{\nu\kappa\alpha} x_{\beta} \int_0^1 du e^{-i(1-u)k \cdot x} \phi(u)$$

$$\phi(1-u) = \phi(u) \Rightarrow \langle 0 | \bar{q}(x) \gamma^{\mu} \frac{\not{1}}{2} q(0) | A(p, \lambda_A) \rangle$$

$$e^{-i p \cdot x} \langle 0 | \bar{q}(-x) \gamma^{\mu} \gamma_5 \frac{\not{1}}{2} q(0) | A(p, \lambda_A) \rangle = F_A^a m_A \int_0^1 du e^{-i(1-u)k \cdot x} \left[k^{\mu} \frac{E \cdot x}{k \cdot x} \phi(u) + E_{\perp}^{\mu} \phi_{\perp}(u) \right]$$

$$\phi(u) = \phi(u)$$

$$\Downarrow \langle 0 | \bar{q}(x) \gamma^{\mu} \frac{\not{1}}{2} q(0) | A(p, \lambda_A) \rangle$$

$$= 2i \int d^4x e^{i q \cdot x} \langle 0 | \bar{q}(x) \mathcal{Q}^2 [g^{\mu\alpha} \gamma^{\nu} + g^{\nu\alpha} \gamma^{\mu} - g^{\mu\nu} \gamma^{\alpha} + i \epsilon^{\mu\nu\alpha\beta} \gamma^{\beta} \gamma_5] q(0) S_{\alpha}^F(x) | A(p, \lambda_A) \rangle$$

$$\mathcal{Q}^2 = a_0 \lambda^0 + a_i \lambda^i \Rightarrow a_0 = \frac{1}{2} \text{Tr}[\lambda^0 \mathcal{Q}^2], \quad a_i = \frac{1}{2} \text{Tr}[\lambda^i \mathcal{Q}^2]$$

$$= 4i \sum_a C_a \int d^4x e^{iq \cdot x} \left\{ i \epsilon^{\mu\nu\alpha\beta} \langle \alpha | \hat{q}(x) | \beta \rangle \frac{x^\alpha}{2} q(\omega) | A(q, \lambda, x) \rangle \right. \\ \left. + \langle \alpha | \hat{q}(x) | \beta \rangle [g^{\mu\alpha} g^{\nu\beta} + g^{\nu\alpha} g^{\mu\beta} - g^{\mu\nu} g^{\alpha\beta}] \frac{x^\alpha}{2} q(\omega) | A(q, \lambda, x) \rangle \right\} S_F^\alpha(x)$$

Inserting the decomposition of the vector and axial-vector matrix elements gives

$$E_{\alpha\mu\nu\beta} = 4i \sum_a C_a F_A^a m_A \int_0^1 du \int d^4x e^{iq \cdot x} \left\{ i \epsilon^{\mu\nu\beta\alpha} S_F^\alpha(x) \right. \\ \times \left[k_\beta \frac{E \cdot x}{k \cdot x} \phi(\omega) + E_{\perp\beta} \phi_{\perp}(\omega) \right] - \frac{1}{4} \epsilon^{\nu\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma\delta} x_\delta S_F^\alpha(x) \phi(\omega) \\ \left. - \frac{1}{4} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma\delta} x_\delta S_F^\alpha(x) \phi(\omega) \right\}$$

?
 $q = q_1 - u p$ because $x^2 = 0$ on LC - but not mentioned in book

$\vec{q} = q_1 - u \vec{k}$
 $= q_1 - u(p - x \frac{m_A^2}{2(p \cdot x)})$

$k_\beta = p_\beta - x_\beta \frac{m_A^2}{2(p \cdot x)}$, $S_F^\alpha(x) \propto x_\alpha$
 $k \cdot x = p \cdot x$, $E_{\perp\beta} = E_\beta - \frac{E \cdot x}{k \cdot x} (k_\beta - \frac{m_A^2}{2(k \cdot x)} x_\beta)$

$$\xrightarrow{q = q_1 - u p, x^2 = 0 \text{ on LC}} 4i \sum_a C_a F_A^a m_A \int_0^1 du \int d^4x e^{iq \cdot x} \left\{ i \epsilon^{\mu\nu\beta\alpha} S_F^\alpha(x) \right. \\ \times \left[p_\beta \frac{E \cdot x}{p \cdot x} (\phi(\omega) - \phi_{\perp}(\omega)) + E_{\beta} \phi_{\perp}(\omega) \right] - \frac{1}{4} \epsilon^{\nu\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma\delta} x_\delta S_F^\alpha(x) \phi(\omega) \\ \left. - \frac{1}{4} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma\delta} x_\delta S_F^\alpha(x) \phi(\omega) \right\}, \quad (*)$$

higher terms

again having neglected \checkmark in the light-cone expansion.

In order to perform the integral, we define

$$\Phi(\omega) = \int dV [\phi(v) - \phi_{\perp}(v)] = \frac{2u-1}{4} \phi(\omega).$$

↑ Mathematica

Using integration by parts, we find

$$\int_0^1 du \int d^4x e^{iq \cdot x} S_F^\alpha(x) \frac{x^\nu}{p \cdot x} [\phi(\omega) - \phi_{\perp}(\omega)] , \quad q = q_1 - u p$$

$$\int_0^1 du e^{iq \cdot x} [\phi(\omega) - \phi_{\perp}(\omega)] = \left(\int_0^1 dv [\phi(v) - \phi_{\perp}(v)] \right) e^{iq \cdot x} \Big|_0^1 \\ = \int_0^1 du \Phi(\omega) [-ix \cdot p] \\ = i \int_0^1 du \Phi(\omega) (p \cdot x)$$

But the operators are LHS, eg. $\hat{q}(x)$ vs \hat{q} have fixed twist namely 2, so how can there be terms of e.g. twist 3 on RHS

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$$= i \int_0^1 du \int d^4x e^{iq \cdot x} S_F^T(x) x^\nu \Phi(u)$$

$$\left\{ \int d^4x x^\mu S_F^\nu(x) e^{iq \cdot x} = \frac{q^\mu}{q^2} - \frac{2q^\mu q^\nu}{q^4}, \text{ as proved before} \right.$$

$$= i \int_0^1 du \Phi(u) \left(\frac{q^{\mu\nu}}{q^2} - \frac{2q^\mu q^\nu}{q^4} \right)$$

The matrix element (*) then becomes

$$E_\alpha M^{\mu\nu\alpha} = 4i \sum_a C_a F_A^a m_A \left\{ i E_\kappa E^{\mu\nu\beta\alpha} P_\beta \int_0^1 du \int d^4x e^{iq \cdot x} S_\alpha^F(x) \right.$$

$$\times \frac{x^\kappa}{p \cdot x} (\phi(u) - \phi_\perp(u)) + i E_\beta E^{\mu\nu\beta\alpha} \int_0^1 du \int d^4x e^{iq \cdot x} S_\alpha^F(x) \phi_\perp(u)$$

$$- \frac{1}{4} E_\alpha E^{\mu\alpha\beta\gamma} P_\beta \int_0^1 du \int d^4x e^{iq \cdot x} x_\delta S_F^\nu(x) \phi(u)$$

$$\left. - \frac{1}{4} E_\alpha E^{\nu\alpha\beta\gamma} P_\beta \int_0^1 du \int d^4x e^{iq \cdot x} x_\delta S_F^\mu(x) \phi(u) \right\}$$

$$= 4i \sum_a C_a F_A^a m_A \left\{ i E_\kappa E^{\mu\nu\beta\alpha} P_\beta \left(i \int_0^1 du \Phi(u) \left[\frac{\delta_\alpha^\kappa}{q^2} - \frac{2q^\alpha q^\kappa}{q^4} \right] \right. \right.$$

$$+ i E_\beta E^{\mu\nu\beta\alpha} \int_0^1 du \left(i \frac{q_\alpha}{q^2} \right) \phi_\perp(u) - \frac{1}{4} E_\alpha E^{\mu\alpha\beta\gamma} P_\beta \int_0^1 du \left(\frac{q_\delta^\nu}{q^2} - \frac{2q_\delta^\nu q^\mu}{q^4} \right)$$

$$\left. \times \phi(u) - \frac{1}{4} E_\alpha E^{\nu\alpha\beta\gamma} P_\beta \int_0^1 du \left(\frac{q_\delta^\mu}{q^2} - \frac{2q_\delta^\mu q^\nu}{q^4} \right) \phi(u) \right\}$$

$$= 4i \sum_a C_a F_A^a m_A E_\alpha \int_0^1 du \left\{ \Phi(u) \left[E^{\alpha\mu\nu\beta} \frac{P_\beta}{q^2} + E^{\mu\nu\beta\alpha} \frac{2P_\beta q_\delta q^\alpha}{q^4} \right] \right.$$

$$\left. - E^{\mu\nu\alpha\beta} \frac{q_\beta}{q^2} \phi_\perp(u) - \frac{1}{4} P_\beta \left[\frac{E^{\mu\alpha\beta\nu}}{q^2} - \frac{2E^{\mu\alpha\beta\gamma} q_\delta q^\nu}{q^4} + \frac{E^{\nu\alpha\beta\mu}}{q^2} \right. \right.$$

$$\left. - \frac{2E^{\mu\alpha\beta\gamma} q_\delta q^\nu}{q^4} \right] \phi(u) \left. \right\}$$

$$q = q_1 - u p = q_1(1-u) - u q_2, \quad p = q_1 + q_2, \quad E(p) \cdot p = 0$$

$$E^{\mu\nu\beta\alpha} P_\beta q_\delta = E^{\mu\nu\beta\alpha} (q_{1\beta} (-u q_{2\delta}) + q_{2\beta} (1-u) q_{1\delta})$$

$$= E^{\mu\nu\beta\alpha} q_{2\beta} q_{1\delta}$$

$$E^{\nu\alpha\beta\mu} P_\beta q_\delta = E^{\nu\alpha\beta\mu} q_{2\beta} q_{1\delta}$$

$$\begin{aligned}
 &= 4i \sum_a C_a F_A^a m_A \epsilon_\alpha \int_0^1 du \left\{ \Phi(u) \left[\epsilon^{\alpha\mu\nu\beta} \frac{p_\beta}{q^2} + \frac{2}{q^4} \epsilon^{\mu\nu\beta\gamma} q_{1\beta} q_{2\gamma} \right] \right. \\
 &\quad \left. - \epsilon^{\alpha\mu\nu\beta} \frac{q_\beta}{q^2} \Phi_\perp(u) + \frac{1}{2q^4} \left[\epsilon^{\alpha\mu\nu\beta} q_{2\beta} q_{1\gamma} q_1^\gamma + \epsilon^{\mu\nu\beta\gamma} q_{2\beta} q_{1\gamma} q_2^\gamma \right] \Phi(u) \right\} \\
 &\quad \left. \epsilon(p) q_1 = -\epsilon(p) q_2 \right. \\
 &= 4i \sum_a C_a F_A^a m_A \epsilon_\alpha \int_0^1 du \left\{ \Phi(u) \left[\epsilon^{\alpha\mu\nu\beta} \frac{p_\beta}{q^2} - \frac{1}{q^4} \epsilon^{\mu\nu\beta\gamma} (q_1 - q_2)^\alpha q_{1\beta} q_{2\gamma} \right] \right. \\
 &\quad \left. - \epsilon^{\alpha\mu\nu\beta} \frac{q_\beta}{q^2} \Phi_\perp(u) + \frac{1}{2q^4} \Phi(u) \left[\epsilon^{\alpha\mu\nu\beta} q_1^\nu q_{1\beta} q_{2\gamma} + \epsilon^{\alpha\mu\nu\beta} q_2^\nu q_{1\beta} q_{2\gamma} \right] \right\}
 \end{aligned}$$

This expression is already gauge invariant, even for non-zero m_A , as readily checked

$$\begin{aligned}
 q_{1\mu} \epsilon_\alpha \mathcal{M}^{\mu\nu\alpha} &= 4i \sum_a C_a F_A^a m_A \epsilon_\alpha \int_0^1 du \left[\Phi(u) \epsilon^{\alpha\mu\nu\beta} \frac{q_{1\mu} q_{2\beta}}{q^2} \right. \\
 &\quad \left. - \epsilon^{\alpha\mu\nu\beta} \frac{q_{1\mu} q_{2\beta}}{q^2} \Phi_\perp(u) + \frac{1}{2q^4} \Phi(u) \epsilon^{\alpha\mu\nu\beta} (q_1 - q_2)^\alpha q_{1\mu} q_{2\beta} \right] \\
 &\stackrel{\epsilon^{\alpha\nu\mu\beta} = -\epsilon^{\alpha\mu\nu\beta}}{\Rightarrow} 4i \sum_a C_a F_A^a m_A \epsilon_\alpha \epsilon^{\alpha\mu\nu\beta} q_{1\mu} q_{2\beta} \int_0^1 du \frac{1}{q^4} \left\{ q^2 (\Phi(u) + u \Phi_\perp(u)) \right. \\
 &\quad \left. - \frac{q_1 \cdot q_2}{2} \Phi(u) \right\}
 \end{aligned}$$

Mathematica $\Rightarrow 0$

$$\text{Alternatively (Mathematica)} \Rightarrow 4i \sum_a C_a F_A^a m_A \epsilon_\alpha \epsilon^{\alpha\mu\nu\beta} q_{1\mu} q_{2\beta} \int_0^1 du \frac{\partial}{\partial u} \left(\frac{3u^2(u-1)}{2q^2} \right)$$

$= 0$

$$\frac{\partial}{\partial u} \left(\frac{3u^2(u-1)}{2q^2} \right) = \frac{6u(u-1) + 3u^2}{2q^2} - \frac{3u^2(u-1)}{2} \frac{\partial}{\partial u} \left(\frac{1}{q^2} \right)$$

$$\left| \frac{\partial}{\partial u} (q^2) = 2q \left(\frac{\partial}{\partial u} q \right) = -2(q \cdot p) = -2q (q_1 + q_2) \right.$$

$$= \frac{3u^2}{2q^2} + \frac{2u}{q^4} \left\{ 3u(u-1) \overset{q_1 \cdot q_2}{q^2} - \frac{3}{2} u^2 (u-1) (-2(q \cdot p)) \right\} = \frac{3u^2}{2q^2} + \frac{1}{q^4} \left\{ 3u(u-1) (q \cdot q_1) \right\}$$

$$\left. \begin{aligned}
 \Phi(u) + u \Phi_\perp(u) &= \frac{2u-1}{4} \Phi(u) + \frac{u}{2} (3-\Phi(u)) = -\frac{\Phi(u)}{4} + \frac{3}{2} u \\
 &= -\frac{3}{2} u(1-u) + \frac{3}{2} u = \frac{3}{2} u^2
 \end{aligned} \right\}$$

How to get here? Dropped term as cov. part to substitution of integral with Mathematica? Deriving the given solution works but how to get there?

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12.10.2020

We can now obtain the form factors by projecting onto the BTT structures - however, we express the matrix element in terms of the $\overline{T}_i^{\mu\nu\alpha}$ that is before applying the Schwinger identities and before dropping unphysical structures. This results in (Mathematica)

$$\sqrt{M^{\mu\nu\alpha}} = 4i \sum_a C_a F_A^a m_A \int_0^1 dx \sum_i C_i \overline{T}_i^{\mu\nu\alpha},$$

$$\text{where } C_3 = \frac{1}{2} (C_1 - C_2 + 3(u-1)u),$$

$$C_4 = \frac{1}{2} (C_1 + C_2),$$

$$C_5 = -C_1 + 3u(u-1)^2 + \frac{3u^2 q_1^2}{2q_1^2} - \frac{3u^2(u-1)(q_1 \cdot q_2)}{q_1^2},$$

$$C_6 = -C_2 + 3u^2(u-1) - \frac{3(u-1)^2 q_2^2}{2q_2^2} - \frac{3(u-1)(q_1 \cdot q_2)}{q_2^2}$$

Using the Schwinger identities

$$\overline{T}_1^{\mu\nu\alpha} = -\frac{1}{2} \overline{T}_3^{\mu\nu\alpha} - \frac{1}{2} \overline{T}_4^{\mu\nu\alpha} + \overline{T}_5^{\mu\nu\alpha}$$

$$\overline{T}_2^{\mu\nu\alpha} = \frac{1}{2} \overline{T}_3^{\mu\nu\alpha} - \frac{1}{2} \overline{T}_4^{\mu\nu\alpha} + \overline{T}_6^{\mu\nu\alpha}$$

and the fact that the third structure projects to zero in any physical quantity, we find

$$C_3 \mapsto \frac{3}{2} (u-1)u \xrightarrow{\text{unphysical}} 0$$

$$C_4 \mapsto 0$$

$$C_5 \mapsto 3u(u-1)^2 + \frac{3u^2 q_1^2}{2q_1^2} - \frac{3u^2(u-1)(q_1 \cdot q_2)}{q_1^2}$$

$$C_6 \mapsto 3u^2(u-1) - \frac{3(u-1)^2 q_2^2}{2q_2^2} - \frac{3u(u-1)(q_1 \cdot q_2)}{q_2^2}$$

Already here, we see that

$$F_1^A(q_1^2, q_2^2) = \mathcal{O}(q_1^{-6}),$$

i.e. the contribution cancels at "this" order.

For $F_{23}^A(q_1^2, q_2^2)$, one has to perform further steps, similar to the scalar case (in paper: "expressing all scalar products in terms of (q_1, q_2) , q_1^2 , and $\frac{\partial}{\partial a} \frac{1}{q_2^2}$, as well as integration by parts."). Recalling the prefactor i/m_A^2 in $\mathcal{M}^{\mu\nu}$, one ultimately finds

$$F_2^A(q_1^2, q_2^2) = 4 \sum_a C_a F_A^a m_A^3 \int_0^1 du \frac{u \phi(u)}{[u q_1^2 + (1-u) q_2^2 - u(1-u) m_A^2]^2},$$

$$F_3^A(q_1^2, q_2^2) = -4 \sum_c C_c F_A^c m_A^3 \int_0^1 du \frac{(1-u) \phi(u)}{[u q_1^2 + (1-u) q_2^2 - u(1-u) m_A^2]^2}.$$

How to bring into form.

The singly-virtual case $F_2(0, q^2)$ (among others) does not converge; this logarithmic end-point singularity has been observed before. Since the found expression $\mathcal{M}^{\mu\nu}$ is gauge invariant and free of kinematic singularities even for finite m_A , it is meaningful to keep the axial-vector mass in the final results.

The "analogous" analysis of tensor mesons is left out here due to many more complications.

We want to summarize the results in terms of their scaling in the average photon virtualities Q^2 and the asymmetry parameter w :

$$Q^2 = \frac{q_1^2 + q_2^2}{2}, \quad w = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}$$

$$\Rightarrow w(q_1^2 + q_2^2) = q_1^2 - q_2^2 \Leftrightarrow q_2^2 = \frac{1-w}{1+w} q_1^2$$

$$\Rightarrow Q^2 = \frac{1}{2} q_1^2 \left(1 + \frac{1-w}{1+w}\right) = \frac{q_1^2}{1+w} \Leftrightarrow q_1^2 = (1+w) Q^2$$

$$\Rightarrow q_2^2 = (1-w) Q^2$$

We start with the pseudoscalar case,

$$F_{PSS}^A(q_1^2, q_2^2) = -4 \sum_a C_a F_P^a \int_0^1 du \frac{\phi(u)}{u q_1^2 + (1-u) q_2^2 - u(1-u) m_P^2}$$

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Why now
again
 $m_p = 0?$

and using Mathematica ($m_p = 0$) find

$$F_{\text{Pxx}}(q_1^2, q_2^2) = -4 \sum_a C_a F_P^a \left\{ \frac{-3(\omega^2 - 1) \log\left(\frac{1-\omega}{1+\omega}\right) + 6\omega}{4Q^2 \omega^3} \right\}$$

$$= -4 \sum_a C_a F_P^a \frac{3}{2\omega^2 Q^2} \left\{ 1 - \frac{\omega^2 - 1}{2\omega} \log \frac{1-\omega}{1+\omega} \right\}$$

$$= \frac{4 \sum_a C_a F_P^a}{Q^2} f^P(\omega)$$

$$f^P(\omega) = -\frac{3}{2\omega^2} \left(1 + \frac{1-\omega^2}{2\omega} \log \frac{1-\omega}{1+\omega} \right)$$

For the scalar case, we had

$$F_1^S(q_1^2, q_2^2) = 4 \sum_a C_a \bar{F}_S^a(\mu) B_1(\mu) m_S \int_0^1 du \frac{3(2u-1)^2 \phi(u)}{uq_1^2 + (1-u)q_2^2}$$

$$F_2^S(q_1^2, q_2^2) = 4 \sum_a C_a \bar{F}_S^a(\mu) B_1(\mu) m_S^3 \int_0^1 du \frac{3u(1-u) \phi(u)}{[uq_1^2 + (1-u)q_2^2]^2}$$

and with Mathematica find

$$F_1^S(q_1^2, q_2^2) = \frac{4 \sum_a C_a \bar{F}_S^a(\mu) B_1(\mu) m_S}{Q^2} f_1^S(\omega)$$

$$F_2^S(q_1^2, q_2^2) = \frac{4 \sum_a C_a \bar{F}_S^a(\mu) B_1(\mu) m_S^3}{Q^4} f_2^S(\omega)$$

$$f_1^S(\omega) = \frac{-9(\omega^2 - 1) \log\left(\frac{1-\omega}{1+\omega}\right) + 6\omega(3 - 2\omega^2)}{4\omega^5}$$

$$= \frac{3}{2\omega^4} \left(3 - 2\omega^2 + 3 \frac{1-\omega^2}{2\omega} \log \frac{1-\omega}{1+\omega} \right)$$

$$f_2^S(\omega) = \frac{-9(\omega^2 - 1) \log\left(\frac{1-\omega}{1+\omega}\right) + 6\omega(3 - 2\omega^2)}{4\omega^5}$$

$$= f_1^S(\omega)$$

Last but not least, we consider the axial-vector meson case, that is

$$F_1^A(q_1^2, q_2^2) = \mathcal{O}(q_i^{-6}) \Rightarrow F_1^A(q_1^2, q_2^2) = \mathcal{O}(\Lambda^{-6})$$

$$F_2^A(q_1^2, q_2^2) = 4 \sum_a C_a F_A^a m_A^3 \int_0^1 du \frac{u \phi(u)}{[u q_1^2 + (1-u) q_2^2 - u(1-u) m_A^2]^2}$$

$$F_3^A(q_1^2, q_2^2) = -4 \sum_a C_a F_A^a m_A^3 \int_0^1 du \frac{(1-u) \phi(u)}{[u q_1^2 + (1-u) q_2^2 - u(1-u) m_A^2]^2}$$

Using Mathematica ($m_A \neq 0$), we obtain

$$F_i^A(q_1^2, q_2^2) = \frac{4 \sum_a C_a F_A^a m_A^3}{\Lambda^4} f_i^A(u) \quad i \in \{2, 3\}$$

$$f_2^A(u) = \frac{-3(u^2 + 2u - 3) \log\left(\frac{1-u}{1+u}\right) + 6u(3-2u)}{8u^4}$$

$$= \frac{3}{4u^3} \left(3 - 2u + \frac{(3+u)(1-u)}{2u} \log \frac{1-u}{1+u} \right)$$

$$f_3^A(u) = -\frac{3(u^2 - 2u - 3) \log\left(\frac{1-u}{1+u}\right) - 6u(3+2u)}{8u^4}$$

Note the odd
(-) sign in
 f_3^A

$$= \frac{3}{4u^3} \left(3 + 2u + \frac{(3-u)(1+u)}{2u} \log \frac{1-u}{1+u} \right)$$

We briefly compare the BLM scaling to "the" quark model approach for the literature. Said model gives the results

$$\frac{F_{\text{BLM}}(q_1^2, q_2^2)}{F_{\text{BLM}}(0,0)} = \frac{m_p^2}{m_p^2 - q_1^2 - q_2^2} \sim \frac{1}{\Lambda^2}$$

$$\frac{F_1^S(q_1^2, q_2^2)}{F_1^S(0,0)} = \frac{m_s^2 (3m_s^2 - q_1^2 - q_2^2)}{3(m_s^2 - q_1^2 - q_2^2)^2} \sim \frac{1}{\Lambda^2}$$

$$\frac{F_2^S(q_1^2, q_2^2)}{F_2^S(0,0)} = \frac{2m_s^4}{3(m_s^2 - q_1^2 - q_2^2)^2} \sim \frac{1}{\Lambda^4} \quad \left(\begin{array}{l} \text{indeed proportional} \\ \text{to normalization} \\ \text{of } F_1^S \end{array} \right)$$

$$F_1^A(q_1^2, q_2^2) = 0$$

$$\frac{F_2^A(q_1^2, q_2^2)}{F_2^A(0,0)} = \frac{F_3^A(q_1^2, q_2^2)}{F_3^A(0,0)} = \left(\frac{m_A^2}{m_A^2 - q_1^2 - q_2^2} \right)^2 \sim \frac{1}{\Lambda^4}$$

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Note that the decay constants are replaced by the TFF normalizations; $F_2^S(q_1^2, q_2^2)$ is indeed proportional to the normalization of F_1^S , because the cross section is assumed to be proportional to the on-shell two-photon width $\Gamma_{\gamma\gamma}$ (or $\tilde{\Gamma}_{\gamma\gamma}$ for axial-vector mesons) in this framework. Moreover, the antisymmetric part of $F_2^A(q_1^2, q_2^2)$ is assumed to vanish, which - apart from the overall sign due to $F_2^A(0,0) = -F_3^A(0,0)$ - makes the two non-zero axial-vector TFFs coincide.

In all cases, the non-vanishing TFFs follow the same asymptotic behavior as given in the BL limits.

We now calculate the symmetric doubly-virtual and singly-virtual form factor F_2^A of the axial-vector mesons. To this end, we define an effective decay constant by

$$F_A^{\text{eff}} = 4 \sum_a C_a F_A^a, \quad C_3 = \frac{1}{6}, \quad C_8 = \frac{1}{6\sqrt{3}}, \quad C_6 = \frac{2}{3\sqrt{6}}$$

So that using Mathematica, we find for

$$F_2^A(q_1^2, q_2^2) = \underbrace{4 \sum_a C_a F_A^a}_{= F_A^{\text{eff}}} \frac{M_A^3}{Q^4} f_2^A(w)$$

$$f_2^A(w) = \frac{3}{4w^3} \left(3 - 2w + \frac{(3+w)(1-w)}{2w} \log \frac{1-w}{1+w} \right)$$

that:	$w = \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}$	$Q^2 = \frac{q_1^2 + q_2^2}{2}$	$F_2^A(q_1^2, q_2^2)$
Sym. doubly-virt. ($q_1^2 = q_2^2 = q^2$)	0	q^2	$\frac{F_A^{\text{eff}} M_A^3}{2q^4} + \mathcal{O}(q^{-6})$
Singly-virtual ($q_1^2 = q^2, q_2^2 = 0$)	1	$q^2/2$	$\frac{3F_A^{\text{eff}} M_A^3}{q^4} + \mathcal{O}(q^{-6})$

Regarding the asymptotic limits

When keeping the axial-vector mass in the integral of $\int_2^A (q_1^2, q_2^2)$, one instead finds

$$F_2^A(q_1^2, q_2^2) = \frac{3F_A^{\text{eff}} m_A^3}{q^4} \times \frac{2}{x^2} \left(\frac{x}{1-x} + \log(1-x) \right), \quad x = \frac{m_A^2}{q^2}$$

I can't continue this? Find something very more complicated that leads to infinity?

Note that additional phenomenological input that could constrain F_A^{eff} is scarce. We can, however, consider these decay constants as they have been estimated using light-cone sum rules (LCSRs), where, in particular, results for $a=0,3,8$ are provided. To extract F_A^{eff} for the physical mesons, mixing effects need to be taken into account. We introduce the mixing angle θ_A via

$$\begin{pmatrix} f_1 \\ f_1' \end{pmatrix} = \begin{pmatrix} \cos\theta_A & \sin\theta_A \\ -\sin\theta_A & \cos\theta_A \end{pmatrix} \begin{pmatrix} f^0 \\ f^8 \end{pmatrix} \iff \begin{pmatrix} f^0 \\ f^8 \end{pmatrix} = \begin{pmatrix} \cos\theta_A & -\sin\theta_A \\ \sin\theta_A & \cos\theta_A \end{pmatrix} \begin{pmatrix} f_1 \\ f_1' \end{pmatrix}$$

From SU(3) symmetry, we have

$$\begin{aligned} \text{Tr}(Q^2 \Phi) &= \frac{1}{9} (3a_1 + 2\sqrt{6} f_1^0 + \sqrt{3} f_1^8) \\ &= \frac{1}{9} (3a_1 + 2\sqrt{6} [\cos\theta_A f_1 - \sin\theta_A f_1'] + \sqrt{3} [\sin\theta_A f_1 + \cos\theta_A f_1']) \\ &= \frac{1}{9} (3a_1 + f_1 [2\sqrt{6} \cos\theta_A + \sqrt{3} \sin\theta_A] + f_1' [-\sqrt{3} \cos\theta_A - 2\sqrt{6} \sin\theta_A]) \end{aligned}$$

So that together with the definition of \tilde{F}_{ss} , we find

$$\frac{\tilde{F}_{\text{ss}}(f_1)}{\tilde{F}_{\text{ss}}(f_1')} \stackrel{\text{Mathematics}}{\downarrow} = \frac{m_{f_1}}{m_{f_1'}} \cot^2(\theta_A - \theta_0), \quad \theta_0 = \arcsin \frac{1}{3} \quad (*)$$

↑ Mixing angle for which two-photon coupling of f_1 vanishes

Likewise, an empirical width for the $a_1(1260)$ can be extracted from SU(3) symmetry:

$$\tilde{\Gamma}_{\text{ss}}(a_1) = \frac{\tilde{F}_{\text{ss}}(f_1)}{3 \cos^2(\theta_A - \theta_0)} \frac{m_{a_1}}{m_{f_1}} = \frac{m_{f_1} \tilde{\Gamma}_{\text{ss}}(f_1') + m_{f_1'} \tilde{\Gamma}_{\text{ss}}(f_1)}{3 m_{f_1} m_{f_1'}} = 2.4 \text{ (ft) keV}$$

insert (*) in RHS to get to LHS.

errors added in quadrature and a generic SU(3) uncertainty

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14.10.2020
 More formal way of doing this? Could write $\langle 0 | \bar{q}(0) \gamma^\mu q(0) | 0 \rangle$ with the current in terms of all flavor matrices and then we are still missing the mixing angle and order of similar? Alternatively because $u\bar{u}$ part of f_1^0 (the one) into bell-like matrices but then the coefficients don't add up? in one of we cited says: produce $F_{f_1^0}$ $\langle 0 | \bar{q} \gamma^\mu q | 0 \rangle$ or $F_{f_1^0}$.

Using $f_i = f_i^0 \cos \theta_A + f_i^8 \sin \theta_A$

e.g. Nucleon's thesis $= \frac{1}{\sqrt{3}} (\bar{u}u + \bar{d}d + \bar{s}s) \cos \theta_A + \frac{1}{\sqrt{6}} (\bar{u}u + \bar{d}d - 2\bar{s}s) \sin \theta_A$
 $\sim \bar{q}(0) \lambda^0 q(0) \quad \sim \bar{q}(0) \lambda^8 q(0)$

$$f_i' = -f_i^0 \sin \theta_A + f_i^8 \cos \theta_A$$

$$= -\frac{1}{\sqrt{3}} (\bar{u}u + \bar{d}d + \bar{s}s) \sin \theta_A + \frac{1}{\sqrt{6}} (\bar{u}u + \bar{d}d - 2\bar{s}s) \sin \theta_A$$

$$a_1 = \frac{1}{\sqrt{2}} (\bar{u}u - \bar{d}d) \sim \bar{q}(0) \lambda^3 q(0)$$

and denoting the decay constants and masses in cartesian basis by F_A^a and m_A^a , we obtain for the decay constants parametrizing the $q = u, d, s$ currents

$$F_{f_i}^u = F_{f_i}^d = F_A^0 \sqrt{\frac{2}{3}} \frac{m_A^0}{m_{f_i}} \cos \theta_A + \frac{F_A^8}{\sqrt{3}} \frac{m_A^8}{m_{f_i}} \sin \theta_A$$

$$F_{f_i}^s = F_A^0 \sqrt{\frac{2}{3}} \frac{m_A^0}{m_{f_i}} \cos \theta_A - \frac{2F_A^8}{\sqrt{3}} \frac{m_A^8}{m_{f_i}} \sin \theta_A$$

$$F_{f_i'}^u = F_{f_i'}^d = -F_A^0 \sqrt{\frac{2}{3}} \frac{m_A^0}{m_{f_i'}} \sin \theta_A + \frac{F_A^8}{\sqrt{3}} \frac{m_A^8}{m_{f_i'}} \cos \theta_A$$

$$F_{f_i'}^s = -F_A^0 \sqrt{\frac{2}{3}} \frac{m_A^0}{m_{f_i'}} \sin \theta_A - \frac{2F_A^8}{\sqrt{3}} \frac{m_A^8}{m_{f_i'}} \cos \theta_A$$

$$F_{a_1}^u = -F_{a_1}^d = F_A^3$$

where we multiplied by an additional factor of $\sqrt{2}$ to had accordance with the paper. Furthermore, isospin symmetry has been assumed and the physical masses of the f_i and f_i' were allowed to differ from the singlet and octet ones.

Ultimately, this leads to

$$F_{f_i}^{\text{eff}} = 2F_A^0 \left(\frac{2}{3}\right)^{3/2} \frac{m_A^0}{m_{f_i}} \cos \theta_A + \frac{2F_A^8}{3\sqrt{3}} \frac{m_A^8}{m_{f_i}} \sin \theta_A$$

$$F_{f_i'}^{\text{eff}} = -2F_A^0 \left(\frac{2}{3}\right)^{3/2} \frac{m_A^0}{m_{f_i'}} \sin \theta_A + \frac{2F_A^8}{3\sqrt{3}} \frac{m_A^8}{m_{f_i'}} \cos \theta_A$$

Where do these relations come from?
 15.11.2020
 Use different basis for $\sum_{i=1}^8 C_i F_i^a$ namely $\sum_{i=1}^8 C_i F_i^a$ are the C_i 's of specific quarks (u, d, s).

where C_i are the C_i 's of specific quarks (u, d, s), $\langle 0 | \bar{q} \gamma^\mu q | 0 \rangle = \frac{1}{\sqrt{3}} (\bar{u}u + \bar{d}d + \bar{s}s) \cos \theta_A + \frac{1}{\sqrt{6}} (\bar{u}u + \bar{d}d - 2\bar{s}s) \sin \theta_A$

$$F_{a_1}^{\text{eff}} = \frac{2}{3} F_A^3$$

From the literature, we have

$$\sqrt{2} F_A^0 = 245(13) \text{ MeV}, \quad \sqrt{2} F_A^8 = 239(13) \text{ MeV}, \quad \sqrt{2} F_A^3 = 238(10) \text{ MeV}$$

$$m_{A^0} = 1,28(6) \text{ GeV}, \quad m_{A^8} = 1,29(5) \text{ GeV},$$

so that

$$F_{\pi^+}^{\text{eff}} = 146(7)(12) \text{ MeV}, \quad F_{\pi^0}^{\text{eff}} = -122(11)(11) \text{ MeV}$$

$$F_{a_1}^{\text{eff}} = 112(5) \text{ MeV}.$$

Using the dipole ansatz

$$F_2^A(q^2, 0) = F_2^A(q^2, 0) \left(1 - \frac{q^2}{\Lambda^2}\right)^{-2}$$

with fit parameters $\tilde{\Gamma}_{\pi^+}(q^2/\Lambda^2) = 3,5(6)(5) \text{ keV} / 3,2(6)(7) \text{ keV}$

$$\Lambda(q^2/\Lambda^2) = 1,04(6)(5) \text{ GeV} / 0,926(72)(31) \text{ GeV}$$

We find that the effective decay constant $F_A^{\text{eff}} = F_2^A(0,0) \frac{m_A}{2}$

as also suggested in the literature - exceeds the above estimate by a factor of about 2. Furthermore, we can extrapolate the dipole fit to find

$$F_2^A(q^2, 0) = \frac{F_2^A(0,0) \Lambda^4}{q^4} = \frac{3 F_A^{\text{eff}} m_A^3}{q^4}$$

$$\Rightarrow F_A^{\text{eff}} = \frac{F_2^A(0,0) \Lambda^4}{3 m_A^3}, \quad \text{so that}$$

$$F_{\pi^+}^{\text{eff}} = 82(26) \text{ MeV}, \quad F_{\pi^0}^{\text{eff}} = -34(12) \text{ MeV},$$

i.e. even lower coefficients. However, in both cases, there is only a single bin above 1 GeV, rendering conclusions about the asymptotics highly uncertain.