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Exercises in Quantum Field Theory, ST 2021

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Additional Exercises

H.X: The Gamma Function

In this exercise, we are going to investigate the Gamma function and its properties. The Gamma function is an indispensable tool for the method of dimensional regularization, which has been introduced in the lecture and will be studied more accessibly in the subsequent exercise.
The Gamma function can be defined via

$$\Gamma(s) \coloneqq \int_0^\infty \mathrm{d}x\, x^{s-1} e^{-x}, \tag{1}$$

which is valid for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

(a) Show that $\Gamma(s+1) = s\Gamma(s)$.

$$\Gamma(s+1) = \int_0^\infty \mathrm{d}x\, x^s e^{-x} \overset{\text{int. by}}{\underset{\text{parts}}{=}} \left[-e^{-x}x^s\right]\Big|_0^\infty + s\int_0^\infty \mathrm{d}x\, x^{s-1}e^{-x} = s\Gamma(s).$$

(b) Show that $\Gamma(s+1) = s!$ for positive integers $s \geq 0$.

Hint: *use the concept of mathematical induction.*

Initial case, $s = 0$:

$$\Gamma(1) = \int_0^\infty \mathrm{d}x\, e^{-x} = \left[-e^{-x}\right]\Big|_0^\infty = 1 = 0!.$$

Assumption for s:

$$\Gamma(s+1) = s!.$$

Induction step, $s \to s+1$:

$$\Gamma(s+2) \overset{(a)}{=} (s+1)\Gamma(s+1) = (s+1)!.$$

(c) Show that $\Gamma(1/2) = \sqrt{\pi}$.

Hint: *the value of the Gaussian integral $\int_{-\infty}^\infty \mathrm{d}x\, e^{-x^2} = \sqrt{\pi}$ might be useful.*

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \mathrm{d}x\, x^{-1/2}e^{-x} \overset{\text{substitute}}{\underset{x=y^2}{=}} \int_0^\infty \mathrm{d}y\,(2y)y^{-1}e^{-y^2} = 2\int_0^\infty \mathrm{d}y\, e^{-y^2}$$
$$\overset{\text{sym. of}}{\underset{\text{integrand}}{=}} \int_{-\infty}^\infty \mathrm{d}y\, e^{-y^2} = \sqrt{\pi}.$$

(d) Let $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$. Show that

$$\alpha^{-s}\Gamma(s) = \int_0^\infty \mathrm{d}x\, x^{s-1} e^{-\alpha x}. \tag{2}$$

<u>Hint</u>: *Prove the identity for $\alpha \in \mathbb{R}$, $\alpha > 0$, and then make use of the identity theorem to analytically continue the obtained equation.*

We let $\alpha \in \mathbb{R}$ with $\alpha > 0$ and start from the right-hand side:

$$\int_0^\infty \mathrm{d}x\, x^{s-1} e^{-\alpha x} \underset{z=\alpha x}{\overset{\text{substitute}}{=}} \int_0^\infty \frac{\mathrm{d}z}{\alpha} \left(\frac{z}{\alpha}\right)^{s-1} e^{-z} = \alpha^{-s} \int_0^\infty \mathrm{d}z\, z^{s-1} e^{-z} = \alpha^{-s}\Gamma(s).$$

Now, if the left-hand and right-hand side of this equation are analytic expressions of $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, we can use the identity theorem to perform the analytic continuation to this domain. Indeed, the right-hand side is analytic in this case because $\alpha^{-s} = 1/\alpha^s$ is analytic except for $\alpha = 0$, which, on the other hand, is due to the fact that the reciprocal of an analytic function is analytic where the function is analytic except for the region where the function is zero. For the left-hand side, we observe that for $\mathrm{Re}(\alpha) > 0$

$$\left| \int_0^\infty \mathrm{d}x\, x^{s-1} e^{-\alpha x} \right| \le \int_0^\infty \mathrm{d}x \left| x^{s-1} e^{-\alpha x} \right| = \int_0^\infty \mathrm{d}x\, x^{s-1} e^{-\mathrm{Re}(\alpha) x} = \mathrm{Re}(\alpha)^{-s}\Gamma(s) < \infty,$$

where we used the initially shown statement again in the last step.

The Gamma function as we defined it in this exercise can be analytically continued to the whole complex plane except for non-positive integers $s = -n \in \{0, -1, -2, \ldots\}$, where it (indeed) possesses poles. This analytic continuation can be performed by using the identity derived in part (a),

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \tag{3}$$

which yields an extension to the domain $\mathrm{Re}(s) < 0$, except for the mentioned poles. More specifically, one can use Eq. (3) one by one, first extending the domain into the half-plane $\mathrm{Re}(s) > -1$ by setting $\Gamma_{-1<\mathrm{Re}(s)<0}(s) = \Gamma(s+1)/s$, then $\Gamma_{-2<\mathrm{Re}(s)<-1}(s) = \Gamma(s+2)/[s(s+1)]$, and so on.

(e) Calculate the value of $\Gamma(-5/2)$ using the analytic continuation defined via Eq. (3). Generalize this result to $\Gamma(-n/2)$ for $n \in \{1, 3, 5, \ldots\}$.

Using the analytic continuation as given above, one finds

$$\begin{aligned}
\Gamma\left(-\frac{1}{2}\right) &= \frac{\Gamma(1/2)}{-1/2} \overset{(c)}{=} -2\sqrt{\pi}, \\
\Gamma\left(-\frac{3}{2}\right) &= \frac{\Gamma(-1/2)}{-3/2} = \frac{4\sqrt{\pi}}{3}, \\
\Gamma\left(-\frac{5}{2}\right) &= \frac{\Gamma(-3/2)}{-5/2} = -\frac{8\sqrt{\pi}}{15}, \\
&\vdots \\
\Gamma\left(-\frac{n}{2}\right) &\underset{m=0,1,2,\ldots}{\overset{n=2m+1}{=}} \Gamma\left(-\frac{2m+1}{2}\right) = \Gamma\left(-m-\frac{1}{2}\right) \\
&= \frac{\Gamma(-m+(m+1/2))}{\underbrace{(-m-1/2)\cdot(-m+1/2)\cdot(-m+3/2)\cdot\ldots\cdot(-1/2)}_{(m+1)\text{ factors}}} \\
&= \frac{(-2)^{m+1}\sqrt{\pi}}{(2m+1)\cdot(2m-1)\cdot(2m-2)\cdot\ldots\cdot 1} = \frac{(-2)^{\frac{n+1}{2}}\sqrt{\pi}}{n\cdot(n-2)\cdot(n-4)\cdot\ldots\cdot 1} \\
&= \frac{(-2)^{\frac{n+1}{2}}\sqrt{\pi}}{n!!}.
\end{aligned}$$

In the following, we are going to investigate the aforementioned pole structure of the Gamma function a little further. To this end, we define the so-called Digamma function as the logarithmic derivative of the Gamma function, i.e.

$$\psi(s) = \frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}. \tag{4}$$

(f) Differentiate the identity derived in part (a) with respect to s to show that

$$\psi(s+1) = \frac{1}{s} + \psi(s). \tag{5}$$

Differentiating

$$s\Gamma(s) = \Gamma(s+1)$$

with respect to s yields

$$\begin{aligned}
&\Gamma(s) + s\Gamma'(s) = \Gamma'(s+1) \\
\Leftrightarrow &1 + \frac{s\Gamma'(s)}{\Gamma(s)} = \frac{\Gamma'(s+1)}{\Gamma(s)} \overset{\text{Eq. (3)}}{=} \frac{s\Gamma'(s+1)}{\Gamma(s+1)},
\end{aligned}$$

so that it follows

$$\psi(s+1) = \frac{\Gamma'(s+1)}{\Gamma(s+1)} = \frac{1}{s} + \frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \psi(s).$$

Restricting Eq. (5) to positive integer values $s = n \in \{0, 1, 2, \ldots\}$ yields the special case

$$\psi(n+1) = \frac{1}{n} + \psi(n) = \frac{1}{n} + \frac{1}{n-1} + \ldots + \psi(1) = -\gamma_{\mathrm{E}} + \sum_{k=1}^{n} \frac{1}{k}, \tag{6}$$

where γ_{E} is the so-called EULER-MASCHERONI constant, defined as $\gamma_{\mathrm{E}} = -\psi(1)$. In order to determine the numerical value of γ_{E}, we now consider STIRLING's formula for the Gamma function,

$$\ln\Gamma(s) = \left(s - \frac{1}{2}\right)\ln(s) - s + \frac{\ln(2\pi)}{2} + \int_0^\infty \mathrm{d}x\, \frac{2\arctan(x/s)}{e^{2\pi x} - 1}, \tag{7}$$

which is valid for $\mathrm{Re}(s) > 0$.[1] An asymptotic expansion of this formula, valid for sufficiently large $s \in \mathbb{R}$, is given by

$$\ln\Gamma(s) = \left(s - \frac{1}{2}\right)\ln(s) - s + \frac{\ln(2\pi)}{2} + \mathcal{O}\left(\frac{1}{s}\right). \tag{8}$$

(g) Show that

$$\gamma_{\mathrm{E}} = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) \tag{9}$$

and convince yourself that $\gamma_{\mathrm{E}} = 0.5772\ldots$ by using a computer algebra system with n sufficiently large, e.g. $n = 100\,000$.[2]

<u>Hint</u>: *consider the asymptotic behavior of Eq.* (6) *for* $n \to \infty$.

Differentiating Eq. (8) with respect to s yields

$$\psi(s) = \ln(s) - \frac{1}{2s} + \mathcal{O}\left(\frac{1}{s^2}\right),$$

so that, in the limit $s \to \infty$, we find $\lim_{s\to\infty}\psi(s) = \ln(s)$, neglecting terms proportional to $\mathcal{O}(1/s)$. Since $\ln(s+1) = \ln(s)$ for asymototic $s \to \infty$, this is equivalent to $\lim_{s\to\infty}\psi(s+1) = \ln(s)$, so that Eq. (6) becomes

$$\gamma_{\mathrm{E}} = \sum_{k=1}^{n}\frac{1}{k} - \psi(n+1) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right).$$

Using *Mathematica* with $n = 500\,000$ yields $\gamma_{\mathrm{E}} = 0.577221$.

We are now in a position to investigate the poles of the Gamma function. For this purpose, we let ϵ be a small real parameter, in particular $\epsilon \to 0$, and expand

$$\Gamma(1+\epsilon) = \Gamma(1) + \epsilon\Gamma'(1) + \mathcal{O}(\epsilon^2) = 1 + \epsilon\Gamma(1)\psi(1) + \mathcal{O}(\epsilon^2) = 1 - \epsilon\gamma_{\mathrm{E}} + \mathcal{O}(\epsilon^2), \tag{10}$$

[1] See, *e.g.*, `https://en.wikipedia.org/wiki/Stirling%27s_approximation` for more information.

[2] Interestingly enough, it is—to the present day—not known whether γ_{E} is an algebraic or transcendental number. In fact, it is not even known whether it is rational or irrational, see, *e.g.*, `https://en.wikipedia.org/wiki/Euler%27s_constant`.

so that using Eq. (3), we find $\Gamma(\epsilon) = \Gamma(1+\epsilon)/\epsilon$ and thus

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_{\mathrm{E}} + \mathcal{O}(\epsilon). \tag{11}$$

(h) Show that for $n > 0$, one has

$$\begin{aligned}\Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right] \\ &= \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} - \gamma_{\mathrm{E}} + 1 + \ldots + \frac{1}{n} + \mathcal{O}(\epsilon)\right].\end{aligned} \tag{12}$$

We prove the formula by induction.
Initial case, $n = 1$:

$$\begin{aligned}\Gamma(-1+\epsilon) &= \frac{\Gamma(\epsilon)}{-1+\epsilon} = -\left[1+\epsilon+\epsilon^2+\mathcal{O}(\epsilon^3)\right]\left[\frac{1}{\epsilon} - \gamma_{\mathrm{E}} + \mathcal{O}(\epsilon)\right] \\ &= -\left[\frac{1}{\epsilon} - \gamma_{\mathrm{E}} + 1 + \mathcal{O}(\epsilon)\right] = -\left[\frac{1}{\epsilon} + \psi(2) + \mathcal{O}(\epsilon)\right].\end{aligned}$$

Assumption for n:

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right].$$

Induction step, $n \to n+1$:

$$\begin{aligned}\Gamma(-(n+1)+\epsilon) &= \frac{\Gamma(-n+\epsilon)}{-n-1+\epsilon} = -\frac{\Gamma(-n+\epsilon)}{n+1-\epsilon} \\ &= -\left[\frac{1}{n+1} + \frac{\epsilon}{(n+1)^2} + \frac{\epsilon^2}{(n+1)^3} + \mathcal{O}(\epsilon^3)\right]\frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right] \\ &= \frac{(-1)^{n+1}}{(n+1)!}\left[\frac{1}{\epsilon} + \psi(n+1) + \frac{1}{n+1} + \mathcal{O}(\epsilon)\right] \\ &= \frac{(-1)^{n+1}}{(n+1)!}\left[\frac{1}{\epsilon} + \psi(n+2) + \mathcal{O}(\epsilon)\right],\end{aligned}$$

where we used that $\psi(n+2) = \psi(n+1) + 1/(n+1)$ in the last step.