Disclaimer

The material at hand was written in the course of a tutoring job at the University of Bonn. If not stated differently in the file or on the following website, the material was prepared solely by me, Marvin Zanke. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given material!

Exercises in Quantum Field Theory, ST 2021

Marvin Zanke

Additional Exercises

H.Y: Dimensional Regularization

In the lecture, we introduced the concept of dimensional regularization, which can be useful when studying integrals that yield an infinite value, in particular in Quantum Field Theory. The purpose of dimensional regularization is to make the divergent behavior of such integrals explicit. In this exercise, we are going to exemplarily investigate some integrals and see how one can deal with them in the framework of dimensional regularization. For simplicity, we work in euclidean space, that is to say you do not have to worry about the MINKOWSKI metric. Besides the Gamma function, which we investigated in detail in an earlier exercise, n-dimensional spherical coordinates are at the foundation of dimensional regularization. For $x \in \mathbb{R}^n$, $n \geq 2$, $x = (x_1, x_2, x_3, \dots, x_{n-2}, x_{n-1}, x_n)^\intercal$, these can be defined via a transformation

$$\begin{aligned} f^{(n)} \colon \Omega_n \mapsto \mathbb{R}^n, \quad x = f^{(n)}(y), \quad y = (r, \phi_1, \phi_2, \dots, \phi_{n-3}, \phi_{n-2}, \phi_{n-1})^\intercal \\ \Omega_n = [0, \infty) \times \underbrace{[0, \pi] \times [0, \pi] \times \ldots \times [0, \pi] \times [0, \pi]}_{(n-2)\text{ times}} \times [0, 2\pi), \end{aligned} \tag{1}$$

according to

$$f^{(n)}(y) = r \begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1)\cos(\phi_2) \\ \sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\ \vdots \\ \sin(\phi_1)\ldots\sin(\phi_{n-3})\cos(\phi_{n-2}) \\ \sin(\phi_1)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1}) \\ \sin(\phi_1)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1}) \end{pmatrix}. \tag{2}$$

(a) Convince yourself that the Jacobian matrix $(J_{(n)})_{ij} = \partial f_i^{(n)}/\partial y_j$ of the transformation from n-dimensional cartesian coordinates to n-dimensional spherical coordinates in the convention of Eq. (2) is given by

$$J_{(n)} = \left(\frac{\partial f^{(n)}(y)}{\partial r} \; \frac{\partial f^{(n)}(y)}{\partial \phi_1} \; \ldots \; \frac{\partial f^{(n)}(y)}{\partial \phi_{n-2}} \; \frac{\partial f^{(n)}(y)}{\partial \phi_{n-1}} \right), \tag{3}$$

with the corresponding vectors $\partial f^{(n)}(y)/\partial(r, \phi_1, \phi_{n-2}, \phi_{n-1})$ given in Tab. 1.

(b) Show that the Jacobian determinant is given by

$$\det(J_{(n)}) = r^{n-1} \prod_{i=2}^{n-1} \sin^{i-1}(\phi_{n-i}). \tag{4}$$

Hint: *repeat part (a) for $n+1$ and use the concept of mathematical induction.*

i	$\left(\partial f^{(n)}(y)/\partial r\right)_i$	$\left(\partial f^{(n)}(y)/\partial \phi_1\right)_i$
1	$\cos(\phi_1)$	$-r\sin(\phi_1)$
2	$\sin(\phi_1)\cos(\phi_2)$	$r\cos(\phi_1)\cos(\phi_2)$
3	$\sin(\phi_1)\sin(\phi_2)\cos(\phi_3)$	$r\cos(\phi_1)\sin(\phi_2)\cos(\phi_3)$
$\vdots$	$\vdots$	$\vdots$
$n-2$	$\sin(\phi_1)\dots\sin(\phi_{n-3})\cos(\phi_{n-2})$	$r\cos(\phi_1)\sin(\phi_2)\dots\sin(\phi_{n-3})\cos(\phi_{n-2})$
$n-1$	$\sin(\phi_1)\dots\sin(\phi_{n-2})\cos(\phi_{n-1})$	$r\cos(\phi_1)\sin(\phi_2)\dots\sin(\phi_{n-2})\cos(\phi_{n-1})$
n	$\sin(\phi_1)\dots\sin(\phi_{n-2})\sin(\phi_{n-1})$	$r\cos(\phi_1)\sin(\phi_2)\dots\sin(\phi_{n-2})\sin(\phi_{n-1})$

i	$\left(\partial f^{(n)}(y)/\partial \phi_{n-2}\right)_i$	$\left(\partial f^{(n)}(y)/\partial \phi_{n-1}\right)_i$
1	0	0
2	0	0
3	0	0
$\vdots$	$\vdots$	$\vdots$
$n-2$	$-r\sin(\phi_1)\dots\sin(\phi_{n-2})$	0
$n-1$	$r\sin(\phi_1)\dots\sin(\phi_{n-3})\cos(\phi_{n-2})\cos(\phi_{n-1})$	$-r\sin(\phi_1)\dots\sin(\phi_{n-2})\sin(\phi_{n-1})$
n	$r\sin(\phi_1)\dots\sin(\phi_{n-3})\cos(\phi_{n-2})\sin(\phi_{n-1})$	$r\sin(\phi_1)\dots\sin(\phi_{n-2})\cos(\phi_{n-1})$

Table 1: Contents of the Jacobian matrix of Eq. (3).

(c) Show that the surface area S_{n-1} of the so-called $(n-1)$-sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, embedded in n-dimensional space, is given by

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}. \tag{5}$$

Hint: *it might be useful to consider the integral* $\int_{-\infty}^{\infty} \mathrm{d}x_1 \dots \int_{-\infty}^{\infty} \mathrm{d}x_n \, e^{-x_1^2-\dots-x_n^2}$, *once in cartesian and once in spherical coordinates, and use the Gaussian integral* $\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2} = \sqrt{\pi}$ *as well as the results obtained in the exercise "The Gamma Function".*

(d) Let now $x \in \mathbb{R}^4$, $a \in \mathbb{R}$, and consider the integral

$$I_4^1(a) = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \frac{1}{x^2+a}. \tag{6}$$

Calculate the indefinite integral and use the result to show that $I_4^1(a)$ diverges.

Hint: *exploit the symmetry of the integrand and use the result(s) shown above.*

(e) Now let $x \in \mathbb{R}$ and consider the integral

$$I_1^1(a) = \int_{\mathbb{R}} \mathrm{d}x \, \frac{1}{x^2+a}.$$

Calculate the indefinite integral and use the result to show that $I_1^1(a) = \pi/(2\sqrt{a})$.

(f) Finally, let us return to the general case $x \in \mathbb{R}^n$ and consider the integral

$$I_n^k(a) = \int_{\mathbb{R}^n} \mathrm{d}^n x \, \frac{1}{(x^2 + a)^k}. \tag{7}$$

Show that

$$I_n^k(a) = \pi^{n/2} a^{n/2-k} \frac{\Gamma(k - n/2)}{\Gamma(k)}. \tag{8}$$

Hint: *make use of the results from the exercise "The Gamma function" and the Gaussian integral* $\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ax^2} = \sqrt{\pi/a}$.

From the properties of the Gamma function, one can thus deduce for which values of n and k the integral diverges/converges. Having regularized an integral, the next step in Quantum Field Theory is the so-called renormalization, which, however, is beyond the scope of this exercise.[1]

[1] Note that in Quantum Field Theory calculations, one usually uses d for the dimension instead of n. The variable n is then instead commonly used for the k introduced in Eq. (7).