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Exercises in Quantum Field Theory, ST 2021

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Additional Exercises

H.Y: Dimensional Regularization

In the lecture, we introduced the concept of dimensional regularization, which can be useful when studying integrals that yield an infinite value, in particular in Quantum Field Theory. The purpose of dimensional regularization is to make the divergent behavior of such integrals explicit. In this exercise, we are going to exemplarily investigate some integrals and see how one can deal with them in the framework of dimensional regularization. For simplicity, we work in euclidean space, that is to say you do not have to worry about the MINKOWSKI metric. Besides the Gamma function, which we investigated in detail in an earlier exercise, n-dimensional spherical coordinates are at the foundation of dimensional regularization. For $x \in \mathbb{R}^n$, $n \geq 2$, $x = (x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n)^\intercal$, these can be defined via a transformation

$$\begin{aligned} f^{(n)} \colon \Omega_n \mapsto \mathbb{R}^n, \quad x = f^{(n)}(y), \quad y = (r, \phi_1, \phi_2, \ldots, \phi_{n-3}, \phi_{n-2}, \phi_{n-1})^\intercal \\ \Omega_n = [0,\infty) \times \underbrace{[0,\pi] \times [0,\pi] \times \ldots \times [0,\pi] \times [0,\pi]}_{(n-2)\text{ times}} \times [0, 2\pi), \end{aligned} \tag{1}$$

according to

$$f^{(n)}(y) = r \begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1)\cos(\phi_2) \\ \sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\ \vdots \\ \sin(\phi_1)\ldots\sin(\phi_{n-3})\cos(\phi_{n-2}) \\ \sin(\phi_1)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1}) \\ \sin(\phi_1)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1}) \end{pmatrix}. \tag{2}$$

(a) Convince yourself that the Jacobian matrix $(J_{(n)})_{ij} = \partial f_i^{(n)}/\partial y_j$ of the transformation from n-dimensional cartesian coordinates to n-dimensional spherical coordinates in the convention of Eq. (2) is given by

$$J_{(n)} = \begin{pmatrix} \frac{\partial f^{(n)}(y)}{\partial r} & \frac{\partial f^{(n)}(y)}{\partial \phi_1} & \ldots & \frac{\partial f^{(n)}(y)}{\partial \phi_{n-2}} & \frac{\partial f^{(n)}(y)}{\partial \phi_{n-1}} \end{pmatrix}, \tag{3}$$

with the corresponding vectors $\partial f^{(n)}(y)/\partial(r, \phi_1, \phi_{n-2}, \phi_{n-1})$ given in Tab. 1.

> The Jacobian matrix in the form of Eq. (3) follows from a straightforward differentiation of Eq.(2) with respect to the spherical coordinates $y = (r, \phi_1, \phi_2, \ldots, \phi_{n-3}, \phi_{n-2}, \phi_{n-1})^\intercal$.

(b) Show that the Jacobian determinant is given by

$$\det(J_{(n)}) = r^{n-1} \prod_{i=2}^{n-1} \sin^{i-1}(\phi_{n-i}). \tag{4}$$

i	$\left(\partial f^{(n)}(y)/\partial r\right)_i$	$\left(\partial f^{(n)}(y)/\partial \phi_1\right)_i$
1	$\cos(\phi_1)$	$-r\sin(\phi_1)$
2	$\sin(\phi_1)\cos(\phi_2)$	$r\cos(\phi_1)\cos(\phi_2)$
3	$\sin(\phi_1)\sin(\phi_2)\cos(\phi_3)$	$r\cos(\phi_1)\sin(\phi_2)\cos(\phi_3)$
$\vdots$	$\vdots$	$\vdots$
$n-2$	$\sin(\phi_1)\ldots\sin(\phi_{n-3})\cos(\phi_{n-2})$	$r\cos(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-3})\cos(\phi_{n-2})$
$n-1$	$\sin(\phi_1)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1})$	$r\cos(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1})$
n	$\sin(\phi_1)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1})$	$r\cos(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1})$

i	$\left(\partial f^{(n)}(y)/\partial \phi_{n-2}\right)_i$	$\left(\partial f^{(n)}(y)/\partial \phi_{n-1}\right)_i$
1	0	0
2	0	0
3	0	0
$\vdots$	$\vdots$	$\vdots$
$n-2$	$-r\sin(\phi_1)\ldots\sin(\phi_{n-2})$	0
$n-1$	$r\sin(\phi_1)\ldots\sin(\phi_{n-3})\cos(\phi_{n-2})\cos(\phi_{n-1})$	$-r\sin(\phi_1)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1})$
n	$r\sin(\phi_1)\ldots\sin(\phi_{n-3})\cos(\phi_{n-2})\sin(\phi_{n-1})$	$r\sin(\phi_1)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1})$

Table 1: Contents of the Jacobian matrix of Eq. (3).

Hint: *repeat part (a) for $n+1$ and use the concept of mathematical induction.*

Repeating part (a) for $n+1$ shows that

$$J_{(n+1)} = \left(\frac{\partial f^{(n+1)}(y)}{\partial r}\ \frac{\partial f^{(n+1)}(y)}{\partial \phi_1}\ \ldots\ \frac{\partial f^{(n+1)}(y)}{\partial \phi_{n-2}}\ \frac{\partial f^{(n+1)}(y)}{\partial \phi_{n-1}}\ \frac{\partial f^{(n+1)}(y)}{\partial \phi_n}\right),$$

where

$$\frac{\partial f^{(n+1)}(y)}{\partial r} = \begin{pmatrix} \partial \widetilde{f}^{(n)}(y)/\partial r \\ \sin(\phi_1)\ldots\sin(\phi_{n-1})\sin(\phi_n) \end{pmatrix},$$

$$\frac{\partial f^{(n+1)}(y)}{\partial \phi_1} = \begin{pmatrix} \partial \widetilde{f}^{(n)}(y)/\partial \phi_1 \\ r\cos(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-1})\sin(\phi_n) \end{pmatrix},$$

$$\frac{\partial f^{(n+1)}(y)}{\partial \phi_{n-2}} = \begin{pmatrix} \partial \widetilde{f}^{(n)}(y)/\partial \phi_{n-2} \\ r\sin(\phi_1)\ldots\sin(\phi_{n-3})\cos(\phi_{n-2})\sin(\phi_{n-1})\sin(\phi_n) \end{pmatrix},$$

$$\frac{\partial f^{(n+1)}(y)}{\partial \phi_{n-1}} = \begin{pmatrix} \partial \widetilde{f}^{(n)}(y)/\partial \phi_{n-1} \\ r\sin(\phi_1)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1})\sin(\phi_n) \end{pmatrix},$$

$$\frac{\partial f^{(n+1)}(y)}{\partial \phi_n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -r\sin(\phi_1)\ldots\sin(\phi_{n-1})\sin(\phi_n) \\ r\sin(\phi_1)\ldots\sin(\phi_{n-1})\cos(\phi_n) \end{pmatrix},$$

and the vectors $\partial\widetilde{f}^{(n)}(y)/\partial y_j$ are obtained from $\partial f^{(n)}(y)/\partial y_j$ by multiplying the last row with $\cos(\phi_n)$, i.e.

$$\frac{\partial\widetilde{f}^{(n)}(y)}{\partial y_j} = \begin{pmatrix} (f^{(n)}(y)/\partial y_j)_1 \\ \vdots \\ (f^{(n)}(y)/\partial y_j)_{n-1} \\ (f^{(n)}(y)/\partial y_j)_n \cos(\phi_n) \end{pmatrix},$$

We can now proceed using mathematical induction.

Initial case, $n = 2$:

The cases $n = 2$ and $n = 3$ are well known to yield $\det(J_{(2)}) = r$ and $\det(J_{(3)}) = r^2 \sin(\phi_1)$ and we keep to ourselves a proof of this result.

Assumption for n:

$$\det(J_{(n)}) = r^{n-1} \prod_{i=2}^{n-1} \sin^{i-1}(\phi_{n-i}).$$

Induction step, $n \to n+1$:

Using the LAPLACE expansion along the last column of $J_{(n+1)}$, we find

$$\begin{aligned}
\det(J_{(n+1)}) &= r\sin(\phi_1)\dots\sin(\phi_{n-1})\cos(\phi_n)\cos(\phi_n)\det(J_{(n)}) \\
&\quad + r\sin(\phi_1)\dots\sin(\phi_{n-1})\sin(\phi_n)\sin(\phi_n)\det(J_{(n)}) \\
&\overset{\sin(x)^2+\cos(x)^2=1}{=} r\sin(\phi_1)\dots\sin(\phi_{n-1})\det(J_{(n)}) \\
&= r\sin(\phi_1)\dots\sin(\phi_{n-1}) r^{n-1} \prod_{i=2}^{n-1} \sin^{i-1}(\phi_{n-i}) \\
&= r^n \prod_{i=2}^{n} \sin^{i-1}(\phi_{n+1-i}),
\end{aligned}$$

where we used that

$$\prod_{i=2}^{n-1} \sin^{i-1}(\phi_{n-i}) = \sin(\phi_{n-2})\sin^2(\phi_{n-3})\sin^3(\phi_{n-4})\dots\sin^{n-3}(\phi_2)\sin^{n-2}(\phi_1),$$

$$\prod_{i=2}^{n} \sin^{i-1}(\phi_{n+1-i}) = \sin(\phi_{n-1})\sin^2(\phi_{n-2})\sin^3(\phi_{n-3})\dots\sin^{n-3}(\phi_3)\sin^{n-2}(\phi_2)\sin^{n-1}(\phi_1).$$

(c) Show that the surface area S_{n-1} of the so-called $(n-1)$-sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, embedded in n-dimensional space, is given by

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}. \tag{5}$$

Hint: *it might be useful to consider the integral* $\int_{-\infty}^{\infty} \mathrm{d}x_1 \dots \int_{-\infty}^{\infty} \mathrm{d}x_n\, e^{-x_1^2-\dots-x_n^2}$*, once in cartesian and once in spherical coordinates, and use the Gaussian integral* $\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2} = \sqrt{\pi}$ *as well as the results obtained in the exercise "The Gamma Function".*

We start by observing that the quantity of interest is given by

$$S_{n-1} = \int_0^{2\pi} \mathrm{d}\phi_{n-1} \int_0^{\pi} \mathrm{d}\phi_{n-2} \ldots \int_0^{\pi} \mathrm{d}\phi_1 \det(J_{(n)}^{r=1}),$$

where $J_{(n)}^{r=1}$ is given by Eq. (4) with $r = 1$. Next, we consider the n-dimensional integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}x_n \, e^{-x_1^2 - \ldots - x_n^2} = \left(\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2} \right)^n = \pi^{n/2},$$

where we used the FUBINI-TONELLI theorem to turn the n-dimensional integral into the n-th power of a one-dimensional integral and made use of the Gaussian integral $\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2} = \sqrt{\pi}$. Alternatively, one can use n-dimensional spherical coordinates to deal with the spherically symmetric integral I, namely

$$\begin{aligned} I &= \int_0^{\infty} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi_{n-1} \int_0^{\pi} \mathrm{d}\phi_{n-2} \ldots \int_0^{\pi} \mathrm{d}\phi_1 \, e^{-r^2} \det(J_{(n)}) \\ &= \int_0^{\infty} \mathrm{d}r \, e^{-r^2} r^{n-1} \underbrace{\int_0^{2\pi} \mathrm{d}\phi_{n-1} \int_0^{\pi} \mathrm{d}\phi_{n-2} \ldots \int_0^{\pi} \mathrm{d}\phi_1 \det(J_{(n)}^{r=1})}_{=S_{n-1}} . \end{aligned}$$

Since

$$\int_0^{\infty} \mathrm{d}r \, e^{-r^2} r^{n-1} \overset{x=r^2}{=} \frac{1}{2} \int_0^{\infty} \mathrm{d}x \, x^{n/2-1} e^{-x} = \frac{\Gamma(n/2)}{2},$$

where we made use of the Gamma function as defined in the exercise "The Gamma function", we find that

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}.$$

Here, we again made use of one of the properties of the Gamma function in the last step.

(d) Let now $x \in \mathbb{R}^4$, $a \in \mathbb{R}$, and consider the integral

$$I_4^1(a) = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \frac{1}{x^2 + a}. \tag{6}$$

Calculate the indefinite integral and use the result to show that $I_4^1(a)$ diverges.

Hint: *exploit the symmetry of the integrand and use the result(s) shown above.*

$$\begin{aligned} I_4^1(a) &= \int_{\mathbb{R}^4} \mathrm{d}^4x \, \frac{1}{x^2 + a} \overset{\text{sph. coord.}}{\underset{\text{and Eq. (5)}}{=}} \frac{4\pi^2}{\Gamma(3)} \int_0^{\infty} \mathrm{d}r \, \frac{r^3}{r^2 + a} = 2\pi^2 \frac{1}{2} \big[r^2 - a \log(a + r^2) \big] \Big|_0^{\infty} \\ &= \pi^2 \big[r^2 - a \log(a + r^2) \big] \Big|_0^{\infty} = \infty. \end{aligned}$$

(e) Now let $x \in \mathbb{R}$ and consider the integral

$$I_1^1(a) = \int_{\mathbb{R}} \mathrm{d}x \, \frac{1}{x^2 + a}.$$

Calculate the indefinite integral and use the result to show that $I_1^1(a) = \pi/(2\sqrt{a})$.

$$I_1^1(a) = \int_{\mathbb{R}} \mathrm{d}x \, \frac{1}{x^2+a} \overset{\text{Mathematica}}{=} \left[\frac{1}{\sqrt{a}} \tan^{-1}\left(\frac{x}{\sqrt{a}} \right) \right] \Bigg|_0^\infty = \frac{\pi}{2\sqrt{a}}.$$

(f) Finally, let us return to the general case $x \in \mathbb{R}^n$ and consider the integral

$$I_n^k(a) = \int_{\mathbb{R}^n} \mathrm{d}^n x \, \frac{1}{(x^2+a)^k}. \tag{7}$$

Show that

$$I_n^k(a) = \pi^{n/2} a^{n/2-k} \frac{\Gamma(k-n/2)}{\Gamma(k)}. \tag{8}$$

Hint: *make use of the results from the exercise "The Gamma function" and the Gaussian integral* $\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ax^2} = \sqrt{\pi/a}$.

$$\begin{aligned} I_n^k(a) &= \int_{\mathbb{R}^n} \mathrm{d}^n x \, \frac{1}{(x^2+a)^k} \overset{(*)}{=} \frac{1}{\Gamma(k)} \int_{\mathbb{R}^n} \mathrm{d}^n x \int_0^\infty \mathrm{d}y \, y^{k-1} e^{-y(x^2+a)} \\ &\underset{\text{integral}}{\overset{\text{Gaussian}}{=}} \frac{\pi^{n/2}}{\Gamma(k)} \int_0^\infty \mathrm{d}y \, y^{k-1-n/2} e^{-ya} \overset{(*)}{=} \pi^{n/2} a^{n/2-k} \frac{\Gamma(k-n/2)}{\Gamma(k)}, \end{aligned}$$

where we used the result of part (d) of the exercise "The Gamma function" in $(*)$. In the second last step, we furthermore swapped the two integrals by virtue of the FUBINI-TONELLI theorem; the justification for using this theorem is given by the fact that the integrands are continuous and assumed integrable over the domain in n dimensions.

From the properties of the Gamma function, one can thus deduce for which values of n and k the integral diverges/converges. Having regularized an integral, the next step in Quantum Field Theory is the so-called renormalization, which, however, is beyond the scope of this exercise.[1]

[1] Note that in Quantum Field Theory calculations, one usually uses d for the dimension instead of n. The variable n is then instead commonly used for the k introduced in Eq. (7).